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A new combinatorial representation of the additive coalescent

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Abstract

The standard additive coalescent starting with n particles is a Markov process which owns several combinatorial representations, one by Pitman as a process of coalescent forests, and one by Chassaing & Louchard as the block sizes in a parking scheme. In the coalescent forest representation, edges are added successively between a random node and a random root. In this paper, we investigate an alternative construction by, instead, adding edges between roots. This construction induces exactly the same process in terms of cluster sizes, meanwhile, it allows us to make numerous new connections with other combinatorial and probabilistic models: size biased percolation, parking scheme in a tree, increasing trees, random cuts of trees. The variety of the combinatorial objects involved justifies our interest in this construction.

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1 Introduction

The aim of this paper is to present a new combinatorial point of view on the additive coalescent starting with n particles with mass 1. In his paper [29], Pitman proposes a combinatorial construction of this process, based on the combinatorics of Cayley trees. In this construction, the coalescence is achieved by adding a sequence of (random) edges in a forest, giving rise eventually to a uniform Cayley tree. Here, we investigate an alternative construction which, in our opinion, leads to a richer combinatorial environment, since it provides links with several other processes defined on trees: a notion of size-biased percolation on a tree, a tree-shaped parking scheme, a random walker model in the tree, a model of decreasing edge-labellings of a Cayley tree, as well as the random cutting problem which dates back to Meir & Moon [26]. Finally, we propose an enriched version of the parking scheme studied by Chassaing & Louchard [13] for the additive coalescence which unifies our construction and Pitman's construction. We also study the asymptotics for the additive coalescence, based on the parking scheme.

Notation. Throughout the paper, for any integer $n \geq 1$, $[n]$ denotes the set $\{1, \dots, n\}$. We use both notation $\#A$ and $|A|$ for the cardinal of a finite set A . If t is a tree or a forest, its size $|t|$ is the number of its nodes. The number of edges of t is denoted by $|t|_E$, and the set of edges by $E(t)$. In the paper, all random variables (r.v.) are assumed to be defined on a common probability space $(\Omega, \mathcal{A}, \mathbb{P})$. For any finite set E , we write $X \sim \text{Uniform}(E)$ to state that the r.v. X is uniform on E .

1.1 Additive coalescent process

Throughout this work, n denotes a positive integer.

Let us first recall the definition of the Marcus–Lushnikov process. At time 0 the system contains n particles with mass 1 labelled from 1 to n . Consider these particles as the vertices of a complete graph with virtual edges, and equip the edges between vertices i and j with random exponential clocks of parameter $K(x_i, x_j) = x_i + x_j$ where x_i and x_j are the masses of i and j . When the clock between i and j rings, replace the masses x_i and x_j by $x_i + x_j$ and 0, and update the parameters of the clocks involving i and j so that the rates remain given by the kernel. This is the so-called Marcus–Lushnikov [25, 24] process, which is a continuous-time Markov process. Marcus–Lushnikov processes may be defined even when the number of particles is infinite, provided that the total mass is finite (see e.g. [16]). The Marcus–Lushnikov process has been introduced in order to model coalescence systems, in which particles have a propensity (represented by the kernel K) to coalesce.

Another representation of this process relies on a combinatorial process taking its values in $\mathcal{P}_{[n]}$, the set of partitions of $[n]$. Again, n particles labelled from 1 to n are present in the system at time 0. The masses do not play an explicit role here, but consider nevertheless that initially all the particles have mass 1. The *additive coalescent* is a Markov chain $\{\Pi_k^+ : 0 \leq k \leq n-1\}$ taking values in $\mathcal{P}_{[n]}$ such that:

- 1) Π_k^+ is a partition of $[n]$ into $n-k$ subsets (so that Π_0^+ is the partition of $[n]$ into n singletons);
- 2) given $\Pi_k^+ = \{\pi_{k,1}^+, \dots, \pi_{k,n-k}^+\}$ with $n_i = |\pi_{k,i}^+|$, $1 \leq i \leq n-k$: Π_{k+1}^+ is obtained from Π_k^+ by merging two parts chosen according to the following distribution : the pair of parts $\{\pi_{k,i}^+, \pi_{k,j}^+\}$ is chosen with probability

$$\frac{n_i + n_j}{\sum_{1 \leq i' < j' \leq n-k} (n_{i'} + n_{j'})} = \frac{n_i + n_j}{n(n-k-1)}, \quad 1 \leq i < j \leq n-k.$$

Next, merge the chosen pair into one subset and let the other parts remain unchanged.

At the level of component sizes, the Markov chain $\{\Pi_k^+ : 0 \leq k \leq n-1\}$ has the same evolution as the Marcus–Lushnikov process at its jumping times.

Proposition 1 ([29], Proposition 6). *For each $0 \leq k \leq n-1$ and each partition $\pi := \{\pi_1, \dots, \pi_{n-k}\}$ of $[n]$ into $n-k$ subsets with $|\pi_i| = n_i$, $1 \leq i \leq n-k$, we have*

$$\mathbb{P}(\Pi_k^+ = \pi) = \frac{\prod_{1 \leq i \leq n-k} n_i^{n_i-1}}{n^k \binom{n-1}{n-k-1}}. \quad (1)$$

An elegant idea to encode the additive coalescent process consists in adding some edges between the components that merge, creating in such a way forests and trees. And this works wonderfully,

since the forests and trees that come into play are particularly simple, and this has numerous implications.

Let us briefly recall Pitman's construction of (Π_k^+) in [29] which is based on a coalescent process of rooted forests. A *rooted forest over $[n]$* here is a graph on $[n]$ whose connected components are trees, each tree t being rooted at one of its vertex r . Here and below, edges in trees are considered directed towards their roots. For $1 \leq k \leq n$, denote by $\mathcal{F}_{k,n}$ the set of rooted forests over $[n]$ which have k tree components. Notice that $\mathcal{F}_{1,n}$ is the set of rooted labelled trees (Cayley trees) with n nodes. By Riordan [31] or Pitman [29, Formula (5)],

$$\#\mathcal{F}_{k,n} = n^{n-k} \binom{n-1}{k-1}, \quad \text{for } 1 \leq k \leq n. \quad (2)$$

Let us define a sequence of random rooted forests $(F_k^{\text{PF}}, 0 \leq k \leq n-1)$ over $[n]$ as follows (the exponent "PF" refers to "Pitman's forest"). First, F_0^{PF} is the only element of $\mathcal{F}_{n,n}$, namely, the trivial forest of n trees reduced to their roots which are labelled from 1 to n .

Pitman's Algorithm (PF Algo).

For $1 \leq k \leq n-1$, to construct F_k^{PF} from F_{k-1}^{PF} , do the following three steps.

- (a) Choose a uniform node α_k^{PF} in F_{k-1}^{PF} . Let t_k be the tree which contains α_k^{PF} .
- (b) Choose a tree t_k^{PF} in F_{k-1}^{PF} uniformly among the trees different from t_k . Denote by β_k^{PF} the root of t_k^{PF} .
- (c) Add the directed edge $\beta_k^{\text{PF}} \rightarrow \alpha_k^{\text{PF}}$.

Clearly, $F_k^{\text{PF}} \in \mathcal{F}_{n-k,n}$ and has exactly k edges, for $0 \leq k \leq n-1$. Observe that the vertex sets of the tree components of F_k^{PF} form a partition of $[n]$; denote by Π_k^{PF} this partition.

Theorem 2. [[29], Theorem 4] (i) The following distribution equality holds

$$(\Pi_k^{\text{PF}}, 0 \leq k \leq n-1) \stackrel{(d)}{=} (\Pi_k^+, 0 \leq k \leq n-1). \quad (3)$$

(ii) Moreover, for $0 \leq k \leq n-1$, $F_k^{\text{PF}} \sim \text{Uniform}(\mathcal{F}_{n-k,n})$.

This representation leads to several interesting consequences:

- one gets a limit representation of the coalescent process when $n \rightarrow +\infty$ (see Section 1.4 for more information);
- the time-reversal of the coalescence process, the so-called fragmentation process, is simple too, since it amounts to choosing at each step a uniform edge and removing it (see Section 3 for details and a related question) ;
- the genealogy of the fragments hence produced has a branching structure, studied per se [10] (see also [9, 14, 12] and Section 3).

We wish to comment on two simple facts:

- It is not apparent at first glance that the evolution of tree sizes in Pitman's construction coincides with the additive coalescent, because of the asymmetry in the construction of the forest. On the

other hand, two trees t_1 and t_2 will be connected when we add an edge either from t_1 to t_2 or from t_2 to t_1 . Thus, the probability that these two trees merge is proportional to $|t_i| + |t_j|$.

• In Pitman's algorithm, an edge is added between a uniform random vertex u taken in the global forest and a random root r taken among the roots of the trees which do not contain u . This rule is somehow arbitrary. Another choice of the edge will yield the same partitions (thus the same distribution as given in Theorem 2 (i)), as long as it connects the same pairs of trees. And a natural question is to wonder what Theorem 2 (ii) will become if another rule is applied instead. The following paragraph answers partially to this question.

1.2 The sorted coalescence algorithm

We propose the following variant of **Pitman's Algo.**

Sorted coalescence algorithm (SC Algo).

Let F_0^{SC} be the trivial forest of n trees. We construct F_k^{SC} from F_{k-1}^{SC} for $1 \leq k \leq n-1$, as follows.

- (a) Choose a uniform node α_k^{SC} in F_{k-1}^{SC} . Let t_k be the tree which contains α_k^{SC} , and let $r(\alpha_k^{\text{SC}})$ be the root of t_k .
- (b) Choose a tree t_k^{SC} in F_{k-1}^{SC} uniformly in F_{k-1}^{SC} among the trees different from t_k . Denote by β_k^{SC} the root of t_k^{SC} .
- (c) Add the directed edge $e_k^{\text{SC}} := r(\alpha_k^{\text{SC}}) \rightarrow \beta_k^{\text{SC}}$.

In other words, the dynamic is almost the same as Pitman's. Instead of adding the edge $\beta_k^{\text{PF}} \rightarrow \alpha_k^{\text{PF}}$, we add the edge $r(\alpha_k^{\text{SC}}) \rightarrow \beta_k^{\text{SC}}$, but the distribution of $(\alpha^{\text{PF}}, \beta^{\text{PF}})$ coincides with that of $(\alpha^{\text{SC}}, \beta^{\text{SC}})$. The new edge is then added between roots of trees in the forest, while it was between a random node and a root before. Since this "small" difference occurs for each k , the forests constructed by both algorithms can be dramatically different.

However in terms of the component size evolution, or in terms of the induced partitions process, both constructions induce exactly the same distribution: at each time step, both constructions consist in merging two components, the first one being chosen with probability proportional to its size, the second one being taken uniformly among the other parts.

For $0 \leq k \leq n-1$, let Π_k^{SC} be the partition of $[n]$ induced by the vertex sets of the tree components of F_k^{SC} . We then have immediately

Theorem 3. *The following distributional equality holds*

$$(\Pi_k^{\text{SC}}, 0 \leq k \leq n-1) \stackrel{(d)}{=} (\Pi_k^{\text{PF}}, 0 \leq k \leq n-1) \stackrel{(d)}{=} (\Pi_k^+, 0 \leq k \leq n-1). \quad (4)$$

On the other hand, by putting the label k on the k th created edge e_k (we denote $L_{e_k}^{\text{SC}} = k$), we obtain at each time step a forest of Cayley trees equipped with a decreasing edge-labelling along the branches (Theorem 6). We call such a tree a *decreasing tree*. Knowing the complete sequence $(F_k^{\text{SC}}, 0 \leq k \leq n-1)$, the edge labelling can be recovered since between time k and $k+1$ only one

edge has been added and it has label $k + 1$. On the other hand, the tree F_{n-1}^{SC} equipped with its edge labelling L_{n-1}^{SC} also encodes completely the history of the additive coalescent.

Remark 4. *Visibly, there is a similarity in the constructions of the processes $(F_k^{\text{PF}}, 0 \leq k \leq n-1)$ and $(F_k^{\text{SC}}, 0 \leq k \leq n-1)$. In Section 2.8, we will construct a coupling between the two sequences of forests.*

1.3 Content of the paper

In Section 2.1, we study the distribution of the process $((F_k^{\text{SC}}, L_k^{\text{SC}}), 0 \leq k \leq n-1)$. In particular, we will see that conditional on F_{n-1}^{SC} , the labelling L_{n-1}^{SC} is uniform among the decreasing edge labellings of F_{n-1}^{SC} (Theorem 6).

We then show that $((F_k^{\text{SC}}, L_k^{\text{SC}}), 0 \leq k \leq n-1)$ – our forest encoding of the additive coalescent – is connected to many new natural stochastic models:

- In Section 2.2, it is shown that given a Cayley tree T , a uniform decreasing edge-labelling may be constructed thanks to a simple Markov model that we call “random walker algorithm” in the tree (Lemma 7).
- In Section 2.3, Proposition 9 shows a link between the uniform decreasing edge-labelling and a percolation process in the tree that we call size-biased percolation: the infection starts from the root; at each step a neighboring node is infected proportionally to the size of the tree hanging out of this node.
- In Section 2.4, the model of decreasing trees is shown to be related with a parking scheme in the tree in which successive cars choose an edge e uniformly and park at the edge nearest to the root among the available edges on the branch from e to the root.

Each of the constructions mentioned above provides a representation of the additive coalescent conditional on the final tree.

- In Section 2.5, we recall the representation of the additive coalescent using a linear parking scheme introduced in Chassaing & Louchard [13]. This scheme allows them to study the asymptotic behaviors of the component sizes in the additive coalescent in the critical window, that is, after $n - \lambda\sqrt{n}$ mergings. In Section 2.6, we revisit and enrich this construction in order to make it encode faithfully the additive coalescent, and not only the component sizes. Using this enriched parking scheme, it is still possible to study its asymptotics and we recover the limit behaviour proved by [13] (see Section 2.9). Moreover, we establish a rather straightforward connection between this enriched parking problems, Pitman construction and our model of decreasing trees (Sections 2.6.2 and 2.6.3).

Finally in Section 3, we use the forests $(F_k^{\text{SC}}, 0 \leq k \leq n-1)$ and the coupling with $(F_k^{\text{PF}}, 0 \leq k \leq n-1)$ to give a new and simple proof on a well-known problem on random cutting of trees.

1.4 Related works on the additive coalescent

The aim of coalescent processes is to model physical/chemical systems containing a large number of weighted particles that have a propensity to merge. There are several types of models, but

a common feature of these models is the use of a collision kernel K : $K(x, y)$ represents the rate at which two particles with masses x and y merge.

Smoluchowski coagulation equations [32, 6] provide a description of the so-called mean-field behavior of a coalescent system with an infinite number of particles by specifying the evolution of the densities of particle masses. These equations form an infinite system of ordinary differential equations. Explicit solutions are available in some special cases, including the additive kernel, i.e., $K(x, y) = x + y$, the multiplicative one, i.e., $K(x, y) = xy$ (see [6, 8] and [18]), and the Kingman coalescent $K(x, y) = 1$ ([21]).

The probabilistic counterpart of the Smoluchowski equations is the Marcus–Lushnikov model [25, 24], intended to describe the continuous-time evolution of the masses – at the particles level – for a system with a finite number of particles. Between each pair of particles (p_1, p_2) , put a clock which rings according to independent exponential variables with parameter $K(\text{Mass}(p_1), \text{Mass}(p_2))$. When a clock rings, merge the corresponding pair of particles and update all the clocks. For any K this is a time homogeneous continuous-time Markov process.

In the additive coalescent case, when starting from n particles with initial mass 1, the time needed to pass from the k th merging to the $(k+1)$ th in the Marcus–Lushnikov model is given by $\tau_k = \min\{\tau(p_1, p_2), \{p_1, p_2\} \in P_k\}$ where the $\tau(p, p')$ are independent, exponentially distributed with parameter $\text{Mass}(p) + \text{Mass}(p')$, and P_k is the set of pairs of (different) particles in the system after k mergings. One checks that τ_k is exponential with parameter $n(n-k-1)$, so that it depends only on the number of mergings already done, and not on the current masses. For this reason, the discrete-time version studied here which is parametrized by the number of mergings is related to the continuous-time version by a simple time change.

There is an extensive literature on general coalescent processes or on the additive coalescent (see Pitman [30, Section 9] for an overview). Let us cite a few works which are most related to the present one.

First of all, in his work [29] where he introduced the model of coalescent forests for the additive coalescence as recalled in Section 1.1, Pitman also observed the following things:

- a) the tree obtained eventually from the construction is uniform among the set of Cayley trees with n nodes;
- b) if we rank the edges according to their appearance time, given the final tree, this ordering of the edges is uniformly distributed among all the edge-orderings of the tree. This leads to a representation of the additive coalescent as a time reversal of a fragmentation process defined on the uniform tree.

We have discussed the case where initially there are n masses. We can also talk about the limit case when $n \rightarrow \infty$. This “limit”, or the convergence of the additive coalescence processes, say, is well understood thanks to Evans & Pitman [16] who showed the following. There exists a Feller process $X^\infty = (X^\infty(t), t \in \mathbb{R}_+)$ taking values in

$$\ell_\downarrow^1 = \{(x_i, i \geq 1) : x_1 \geq \cdots \geq x_i \geq x_{i+1} \geq \cdots \geq 0, \sum x_i < +\infty\}, \quad (5)$$

which has the dynamics of the additive coalescent; moreover, if X^n denotes the Markov process with the same kernel starting from $(1/n, \dots, 1/n, 0, 0, \dots)$, which corresponds to the component

sizes in Marcus–Lushnikov model with n initial masses $1/n$, then

$$X^n(\cdot + (\log n)/2) \xrightarrow[n \rightarrow \infty]{(d)} X^\infty,$$

where the convergence holds in $\mathbb{D}((-\infty, +\infty), \ell_+^1)$. The limit is called the *standard additive coalescent* (see Section 2.9 below for more information). Moreover, the coalescence-fragmentation duality mentioned above remains valid when $n \rightarrow +\infty$, leading to the Aldous–Pitman’s representation [5] of the standard additive coalescent as a time-reversal of a fragmentation process on Aldous’ continuum random tree.

Another combinatorial representation of the additive coalescent has been investigated by Chassaing & Louchard [13] (see Section 2.5). They observe that in a certain parking scheme, known to be related to a well-known data structure named “hashing with linear probing” in computer science, parking blocks evolve as the additive coalescent. This representation has the following advantage: it allows for a direct encoding of the additive coalescent (starting with n particles with mass 1) by a Markov chain taking values in \mathbb{Z} such that the sizes of the parking blocks at each time correspond to the excursion lengths of the Markov chain above its running infimum (see Section 2.5). Moreover, it is shown in [13] that these excursions lengths converge to those of a Brownian excursion with a linear drift. See Bertoin [7] for a related work. Let us also mention that this is close in spirit in the way that Aldous encodes the multiplicative coalescent using another random walk which converges to the Brownian motion with a parabolic drift (see [4]). Recently, Broutin and the first author [11] revisited the study of the so-called standard versions of additive and multiplicative coalescent processes. In particular, relying upon a weak convergence argument, they prove by a unified approach the descriptions of both processes by certain Brownian excursions.

2 A new point of view on the additive coalescent

2.1 Alternative to Pitman’s construction

We discuss here the properties of the sequence of forests $(F_k^{\text{SC}}, 0 \leq k \leq n-1)$ produced by **SC Algo**, introduced in Section 1.2.

The terminology “sorted coalescence algorithm” is motivated by the following fact. We will see that if we fix the final tree, then the coalescence takes place in an ordered manner: along the branches of the tree, from the leaves towards the root.

Recall that the edges of the forest F_k^{SC} have been labelled by their ranks of appearance. We say that an edge-labelling $L = (L_e, e \in E(f))$ of a forest $f = (f_1, \dots, f_k)$ is *decreasing* if it has the following properties:

- the set of labels is $\{1, \dots, |f|_E\}$, each label being used exactly once,
- in each tree f_j , the labelling is decreasing along each simple path going out of its root r : if $(u_0 = r, u_1, \dots, u_h)$ is a simple path in f_j for some $h \geq 1$, then

$$L_{(u_0, u_1)} > \dots > L_{(u_{h-1}, u_h)}. \quad (6)$$

Let $\text{DL}(f)$ be the set of decreasing edge-labellings of a forest f . We make the convention that if the edge set $E(f) = \emptyset$, then $\text{DL}(f) = 1$. For a vertex $u \in f$, let $\text{Sub}_u(f)$ denote the subtree of f rooted at u . Also recall that for a tree or a forest t , $|t|$ stands for the number of its nodes.

Lemma 5. For $1 \leq k \leq n$ and $f \in \mathcal{F}_{k,n}$,

$$\#\text{DL}(f) = (n-k)! \prod_{u \in f \setminus \{\rho_i, 1 \leq i \leq k\}} \frac{1}{|\text{Sub}_u(f)|}, \quad (7)$$

where ρ_i , $1 \leq i \leq k$, are the k roots of f .

Proof. We first show that for a rooted tree t ,

$$\#\text{DL}(t) = \prod_{u \in t} \frac{(|\text{Sub}_u(t)| - 1)!}{\prod_{v: \text{child of } u} |S_v(t)|!}, \quad (8)$$

with the convention that $0! = 1$. Notice that after simplification, (8) leads to the formula (7) when $k = 1$.

Now, (8) can be checked by induction on the size of t : for $|t| = 1$, this is trivial. Suppose (8) holds for trees with sizes up to $m - 1 \geq 1$. For a rooted tree t of size m with root r , denote by J the in-degree of r . Let $(E_j, 1 \leq j \leq J)$ be the partition of the edge set of t induced by the J subtrees above r . Then a decreasing edge labelling of t is obtained by first choosing the subsets of the labels for E_j , $1 \leq j \leq J$, and then assigning the largest label of each subset to the edge adjacent to the root. Applying the induction hypothesis to the labelling of subtrees yields (8). To get to the enumeration for rooted forests, we first choose the subsets of labels for t_i , $1 \leq i \leq k$, then apply (8) for each t_i . \square

Write $T^{\text{SC}} = F_{n-1}^{\text{SC}}$, the final tree obtained from **SC Algo**. Recall that we have introduced an edge labelling by defining $L_e^{\text{SC}} = k$ if e is the k th added edge. Denote by $L^{\text{SC}} = (L_e, e \in E(T^{\text{SC}}))$. For $0 \leq k \leq n - 1$, let L_k^{SC} be the subset of L^{SC} obtained by removing those labels $> k$. Then it is clear that L_k^{SC} is a decreasing edge-labelling of F_k^{SC} . As said before, the pair $(T^{\text{SC}}, L^{\text{SC}})$ encodes the history of the coalescence so that we can recover from $(T^{\text{SC}}, L^{\text{SC}})$ the sequence of the pairs $(r(\alpha_k^{\text{SC}}), \beta_k^{\text{SC}})$, $1 \leq k \leq n - 1$.

Theorem 6. For $0 \leq k \leq n - 1$, we have

$$\mathbb{P}\left((F_k^{\text{SC}}, L_k^{\text{SC}}) = (f, \ell)\right) = \frac{\mathbf{1}_{\mathcal{F}_{n-k,n}}(f)}{\#\mathcal{F}_{n-k,n}} \frac{\mathbf{1}_{\text{DL}(f)}(\ell)}{\#\text{DL}(f)}. \quad (9)$$

Hence, $F_k^{\text{SC}} \sim \text{Uniform}(\mathcal{F}_{n-k,n})$ and conditional on F_k^{SC} , $L_k^{\text{SC}} \sim \text{Uniform}(\text{DL}(F_k^{\text{SC}}))$.

The fact that F_{n-1}^{SC} is uniformly distributed, as is F_{n-1}^{PF} , is perhaps unexpected since the sorted coalescence construction seems to “favor” the neighborhoods of the roots. On the other hand, notice that after the addition of the edge $r(\alpha_k^{\text{SC}}) \rightarrow \beta_k^{\text{SC}}$ $r(\alpha_k^{\text{SC}})$ is not a root anymore, so that in the sequel of the construction, no more edges will be added from or to $r(\alpha_k^{\text{SC}})$.

Proof. Let $f \in \mathcal{F}_{n-k,n}$ and $\ell \in \text{DL}(f)$. Notice that if $(F_k^{\text{SC}}, L_k^{\text{SC}}) = (f, \ell)$ is given we can recover the sequence F_j^{SC} for any $0 \leq j \leq k$ by removing the edges with labels $> j$. In particular, if we write $x_j \rightarrow y_j$ for the edge labelled j in the labelling ℓ (recall that the edges are directed

towards the root), then $\text{Sub}_{x_j}(f)$ is the tree component of F_{j-1}^{SC} which contains α_j^{SC} , $1 \leq j \leq k$. By our construction, the probability that F_j^{SC} is obtained from F_{j-1}^{SC} by adding the edge $x_j \rightarrow y_j$ is $|\text{Sub}_{x_j}(f)|/(n(n-j))$. This yields

$$\mathbb{P}\left(F_k^{\text{SC}} = f; L_k^{\text{SC}} = \ell\right) = \frac{(n-k-1)!}{n^k(n-1)!} \prod_{1 \leq j \leq k} |\text{Sub}_{x_j}(f)|.$$

Since each vertex of f except the $n-k$ roots appears exactly once in $\{x_j, 1 \leq j \leq k\}$, we obtain that

$$\mathbb{P}\left(F_k^{\text{SC}} = f; L_k^{\text{SC}} = \ell\right) = \frac{(n-k-1)!}{n^k(n-1)!} \prod_{u \in f \setminus \{\rho_i, 1 \leq i \leq n-k\}} |\text{Sub}_u(f)|,$$

where ρ_i , $1 \leq i \leq n-k$ are the $n-k$ roots of f . Comparing this with (2) and (7), we get (9). \square

2.2 Random walker algorithm

Here we discuss a model of random walkers in a Cayley tree T with n nodes. In this model, $n-1$ random walkers will climb successively in the tree, walking from one edge to an adjacent one. The k th walker will decide to stop at a certain edge e , and will label this edge k . The global labelling will be a uniform decreasing edge-labelling of T .

At time 0 all the edges of T are white. Let r be the root of T . Add to T an extra white edge e^* from an extra vertex u and r (see Fig. 1). The starting point for the random walkers will always be this edge e^* . Consider $T \cup \{u\}$ rooted at the extra edge e^* . The usual parent-child relation among the nodes can be extended to the edges in an obvious way. This leads to a notion of child-edge of an edge $e \in E(T)$.

Random walker algorithm.

For k going from 1 to $n-1$, the k th walker starts from the root-edge e^ .*

Its journey runs as follows. Assume that at some moment, the walker is at some edge e . Let d be the number of white child-edges of e .

– If $d = 0$, that is, if there are no white child-edges of e , paint in black this last edge e and label it with the rank of the walker (put $L_e^{\text{RW}}(T) = 1$ for the first random walker, $L_e^{\text{RW}}(T) = 2$ for the second, etc).

– if $d \geq 1$, let e_1, e_2, \dots, e_d be the white child-edges of e . The walker goes to e_i with probability proportional to W_i , the number of the white edges contained in the subtree of T rooted at e_i (including e_i).

Hence, when k walkers have climbed, there are exactly k black edges, labelled from 1 to k (see Fig. 1). In the end, all the edges of T are painted black. Moreover, T is equipped with an edge-labelling $L^{\text{RW}}(T) = (L_e^{\text{RW}}(T), e \in E(T))$ induced by the ranks of the walkers.

Lemma 7. *For each fixed $T \in \mathcal{F}_{1,n}$, $L^{\text{RW}}(T) \sim \text{Uniform}(\text{DL}(T))$.*

Proof. If e_i , $1 \leq i \leq k$, are the edges adjacent to the edge-root e^* , write T_i^+ for the subtrees rooted at e_i . By a decomposition at the root and an induction argument, we see that the proof

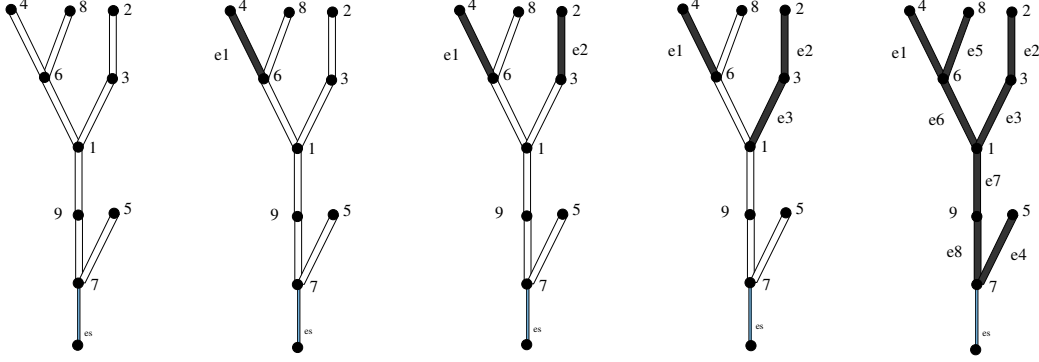


Figure 1: An example of the progressive edge labelling by the random walkers. The first edge had probability $(7/8)(3/5)(1/2) = 21/80$ to be chosen, after this choice, the second one had probability $(6/7)(2/4) = 12/28$ to be chosen. In terms of partitions, the evolution is as follows: $\Pi_0^{\text{RW}} = \{\{1\}, \dots, \{9\}\}$, $\Pi_1^{\text{RW}} = \{\{1\}, \{2\}, \{3\}, \{4, 6\}, \{5\}, \{7\}, \{8\}, \{9\}\}$, $\Pi_2^{\text{RW}} = \{\{1\}, \{2, 3\}, \{4, 6\}, \{5\}, \{7\}, \{8\}, \{9\}\}$, $\Pi_3^{\text{RW}} = \{\{1, 2, 3\}, \{4, 6\}, \{5\}, \{7\}, \{8\}, \{9\}\}$, $\Pi_8^{\text{RW}} = \{1, \dots, 9\}$.

reduces to identifying the joint distribution of $(\{\ell(e), e \in T_i^+, 1 \leq i \leq k\})$, the sets of the labels attributed to the T_i^+ 's. For this, only the first step of each walker needs to be followed. If the w th walker goes in T_i^+ , let us set $c(w) = i$, $1 \leq w \leq n - 1$. When t walkers have climbed, let $l_t(i) = \#\{w \in \{1, 2, \dots, t\} : c(w) = i\}$ be the number of the walkers gone into T_i^+ ; then $\mathbb{P}(c(t+1) = i) = (|T_i^+|_E - l_t(i)) / (|T|_E - t)$. It follows that $(\{\ell(e), e \in T_i^+, 1 \leq i \leq k\})$ forms a uniform partition of $\{1, \dots, |T|_E\}$ into k parts of respective sizes $|T_i^+|_E$, $1 \leq i \leq k$. By the previous arguments, this shows that $L^{\text{RW}}(T) \sim \text{Uniform}(\text{DL}(T))$. \square

For $0 \leq k \leq n - 1$, by discarding the edges e whose labels $L_e^{\text{RW}}(T) > k$, we obtain a forest over $[n]$. Then the vertex sets of the tree components of this forest induce a partition of $[n]$, which we denote by Π_k^{RW} . Alternatively, Π_k^{RW} can be defined by looking at the black components of T defined as follows. When k walkers have climbed, there are exactly k black edges. Then we find in T (see Fig. 1):

- the white edges constitute the edge set of a subtree of T which contains the root,
- the black edges with the n vertices form a forest over $[n]$ with $n - k$ black tree components.

Here is another representation of the additive coalescent:

Theorem 8. For $T \sim \text{Uniform}(\mathcal{F}_{1,n})$,

$$\left(\Pi_k^{\text{RW}}, 0 \leq k \leq n - 1\right) \stackrel{(d)}{=} \left(\Pi_k^+, 0 \leq k \leq n - 1\right).$$

Proof. From Lemma 7 and Theorems 3 and 6, we readily see that the pair $(T, L^{\text{RW}}(T))$ produced by the random walkers model has the same distribution as $(T^{\text{SC}}, L^{\text{SC}})$. \square

2.3 Size-biased percolation

We introduce a model of percolation on a tree that we call *size-biased percolation*. We will see that it is a time-reversal of the random walker process. Take a tree T rooted at r in $\mathcal{F}_{1,n}$. We define a sequence $(\text{Perc}_k(T), 0 \leq k \leq |T|_E)$ of subtrees of T all containing the root r , which will represent the percolated cluster at each time.

- At time $k = 0$, the cluster $\text{Perc}_k(T)$ is reduced to $\{r\}$.
- For $0 \leq k \leq |T|_E - 1$, choose a random edge among those adjacent to $\text{Perc}_k(T)$ with the following distribution: an edge $e = a \rightarrow b$ with $b \in \text{Perc}_k(T)$ and $a \notin \text{Perc}_k(T)$ is chosen with probability proportional to $|\text{Sub}_a(T)|$.

Observe that $k \mapsto \text{Perc}_k(T)$ is increasing for the inclusion order. For $0 \leq k \leq |T|_E$, let T_k^{Perc} be the graph obtained from T by removing the edges of $\text{Perc}_k(T)$ (the nodes are kept), and let $\Pi_k^{\text{Perc}(T)}$ be the partition of $[n]$ induced by the vertex sets of the components of T_k^{Perc} (see Fig. 2).

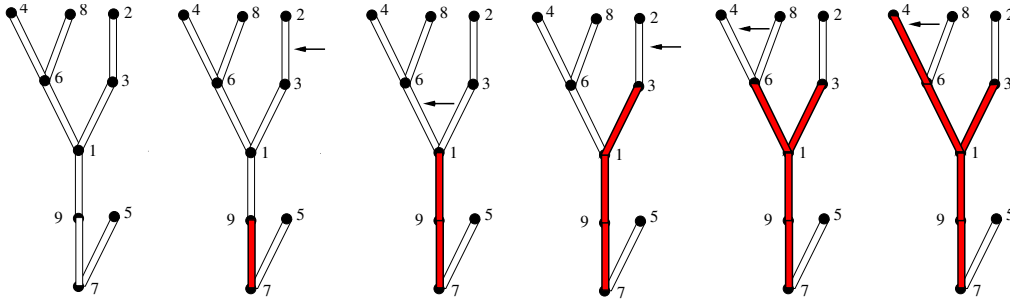


Figure 2: construction of $(\text{Perc}_k(T), 0 \leq k \leq |T|_E)$ by successively adding edges to the present cluster (drawn in red). Each time, choose a uniform white edge e and color in red the white edge on the path from e to the root which is nearest to the root.

As before, we introduce an edge-labelling $L^{\text{Perc}}(T)$ of T by putting a label $n - k$ to the unique edge e in $\text{Perc}_k(T) \setminus \text{Perc}_{k-1}(T)$, $1 \leq k \leq |T|_E$. It is easy to see that $L^{\text{Perc}}(T)$ belongs to $\text{DL}(T)$, the set of decreasing edge-labellings of T .

Proposition 9. *For any rooted tree T we have $L^{\text{Perc}}(T) \sim \text{Uniform}(\text{DL}(T))$. If T is uniform in $\mathcal{F}_{1,n}$, then*

$$\left(\Pi_{n-1-k}^{\text{Perc}(T)}, 0 \leq k \leq |T|_E \right) \stackrel{(d)}{=} \left(\Pi_k^{\text{SC}}, 0 \leq k \leq |T|_E \right); \quad (10)$$

as a consequence, $\left(\Pi_{n-1-k}^{\text{Perc}(T)}, 0 \leq k \leq n-1 \right)$ is the additive coalescent.

Proof. The proof is similar to the one of Lemma 7; we omit the details. \square

2.4 A tree-shaped parking scheme

We claim that for any fixed $T \in \mathcal{F}_{1,n}$, the process $(\text{Perc}_k(T), 0 \leq k \leq n-1)$ may be represented as a tree-shaped parking scheme (TSPS). In the usual “linear” parking scheme, the places are

arranged along the line (see Section 2.5). The cars successively choose a place c and then park at the first free place among $c, c+1, c+2, \dots$. In the tree-shaped parking scheme, the cars choose a place c and then parks at the first free place on the path from the root to c . More precisely, there are two equivalent versions:

- A) In this version, a place refers to a node. Consider a sequence $(V_i, i \geq 1)$ of i.i.d. random variables uniformly distributed on the set of nodes of T . We will rely on this sequence to define the places occupied by the successive cars. Set $k_0 = 0$ and $\text{TPS}_0(T) = \{r\}$ (the tree reduced to the root); for $1 \leq j \leq n-1$, let $k_j = \min\{i > k_{j-1} : V_i \notin \text{TPS}_{j-1}(T)\}$ (the first chosen place out of $\text{TPS}_{j-1}(T)$) and denote by V'_j the node on the path $\llbracket r, V_{k_j} \rrbracket$ which is nearest to r and not contained in $\text{TPS}_{j-1}(T)$; then define $\text{TPS}_j(T)$ as the smallest tree containing $\text{TPS}_{j-1}(T)$ and V'_j . The nodes of the subtree $\text{TPS}_j(T)$ other than r represent the places taken by the first j cars. Clearly, $(\text{TPS}_j(T), 0 \leq j \leq n-1)$ is an increasing sequence of subtrees of T .
- B) In this second version, a place is an edge instead of a node. Start from a tree T whose edges are painted in white. Each time, one picks uniformly at random a white edge e in T , and color in red the last white edge on the path from e to the root. In this way, we obtain a sequence of subtrees formed by the red edges (see Fig. 2). It is clear that this sequence has the same distribution as $(\text{TPS}_j(T), 0 \leq j \leq n-1)$ as defined above.

Once again, one may attribute a labelling $L^{\text{TPS}}(T)$ to the edges according to their entrance in $\text{TPS}(T)$: put a label $L_e^{\text{TPS}}(T) = n - k$ to the k th added edge. This labelling belongs to $\text{DL}(T)$. Moreover, we have the following.

Proposition 10. *For any rooted tree T , the following identities in distribution hold*

$$L^{\text{TPS}}(T) \stackrel{(d)}{=} L^{\text{Perc}}(T); \quad (11)$$

$$(\text{TPS}_k(T), 0 \leq k \leq |T|_E) \stackrel{(d)}{=} (\text{Perc}_k(T), 0 \leq k \leq |T|_E). \quad (12)$$

Proof. To choose a subtree τ of $T \setminus \text{Perc}_k(T)$ with probability proportional to its size, one can sample a sequence of i.i.d. uniform variables $(U_i, i \geq 1)$ taking values in the vertex set of T until the first moment that some U_i falls in $T \setminus \text{Perc}_k(T)$. The subtree τ of $T \setminus \text{Perc}_k(T)$ which contains this first U_i has the right distribution. \square

2.5 Linear parking scheme representation and its asymptotics

Here, in contrast with Section 2.4, parking scheme refers to linear/circular parking scheme. Let us explain some of the considerations introduced by Chassaing & Louchard [13] before proposing an enriched version of their construction, which will allow us to unify the parking representation with the model of coalescent forests of Pitman and our decreasing trees. The parking contains n places labelled from 1 to n . The parking is circular so that place $n+1$ and 1 coincide: Here, for $x \in \mathbb{Z}$, \bar{x} stands for the integer in $[n]$ equaling x modulo n .

The parking scheme can be presented as an algorithm : a vector $(C_i, 1 \leq i \leq n-1)$ taking values in $[n]^{n-1}$ is entered as data, the algorithm then produces a sequence of occupations $(P_i)_{1 \leq i \leq n-1}$.

In the beginning, all places are free. There will be a total number of $n - 1$ cars which will come to park.

Parking algorithm.

At time i , $1 \leq i \leq n - 1$, Car i enters the parking and chooses the place C_i .

Upon its arrival, Car i parks at P_i , the first free place among $\overline{C_i}, \overline{C_i + 1}, \overline{C_i + 2}, \dots$.

We may reformulate the previous condition as follows : Car i parks at $P_i := \overline{C_i + l_*}$, where $l_* := \min\{l \in \{0, 1, \dots, n - 1\} : \overline{C_i + l} \notin \{P_1, \dots, P_{i-1}\}\}$.

In the sequel we will consider the case in which the C_1, \dots, C_{n-1} are random variables, i.i.d. and uniform in $[n]$.

Note 11. It will be useful to consider the following construction of the random variables C'_i 's on the probability space $(\Omega^{\text{Par}}, \mathcal{A}^{\text{Par}}, \mathbb{P}^{\text{Par}})$ where $\Omega^{\text{Par}} = [n]^{n-1}$, $\mathcal{A}^{\text{Par}} = \text{Powerset}(\Omega^{\text{Par}})$, and \mathbb{P}^{Par} , the uniform distribution on Ω^{Par} . On this space, the functions C_i are defined by $C_i(\omega) = \pi_i(\omega)$ (the extraction of the i th coordinate). The parking process $(P_i, 1 \leq i \leq n - 1)$, as well as the quantities that will be defined in this section, are random variables on this space. In this case, computations on probabilities often reduce to enumeration of the related subsets of Ω . Since the results stated below only concern the distribution of C'_i 's, we will assume without loss of generality that these random variables are constructed in this way.

For any $i, j \in [n]$, denote by $\overline{[i, j]}$ the circular interval formed by $\overline{i}, \overline{i + 1}, \dots, \overline{i + k}$ where $k = j - i$.

Define a *block* in the parking as the maximal circular interval $\overline{[x, x + k]}$ (maximal for the inclusion order) satisfying that $\overline{x + i}$ is occupied for $0 \leq i < k$, and $\overline{x + k}$ is not occupied. With this definition, we have the following description of the parking.

- In the beginning, the blocks are the n trivial blocks $I[0] := (I_j^0, 1 \leq j \leq n)$ where $I_j^0 = \overline{[j, j]}$;
- At all times, the number of blocks is given by the number of empty places. Thus, for $0 \leq m \leq n - 1$, when m cars have parked, there are exactly $n - m$ blocks which we denote by $I[m] := (I_j^m, 1 \leq j \leq n - m)$. Notice that $I[m]$ forms a partition of $\overline{[1, n]}$: the total block sizes sum to n for any m , since at each step an empty place is replaced by an occupied one. Write now $|I[m]|$ for the multiset $\{|I_j^m|, 1 \leq j \leq n - m\}$.

There is some slight difference between the above definition and the original one in [13]. Namely, here the block size is 1 plus the number of the cars in it. Thanks to this small modification, we have an exact correspondence between the parking scheme and the additive coalescent (this point is also discussed in [13, Section 8].) To state the correspondence, we need the following notation. For a partition $\pi = (\pi_1, \dots, \pi_k)$ of $[n]$, denote by $|\pi|$ the multiset $\{|\pi_j|, 1 \leq j \leq k\}$.

Proposition 12. [Chassaing & Louchard [13]] *The block sizes in the parking scheme evolve in the same way as the cluster sizes in the additive coalescence:*

$$(|I[k]|, 0 \leq k \leq n - 1) \stackrel{(d)}{=} (|\Pi_k^+|, 0 \leq k \leq n - 1). \quad (13)$$

Let us mention that the proof we propose for Proposition 17 also uses ideas from Chassaing & Louchard [13].

Remark 13. *An alternative description for the evolution of the block sizes can be given as follows. For m varying from 0 to $n - 2$,*

- *let $j \in \{1, \dots, n - m\}$ be the index of the block containing C_{m+1} (that is, $C_{m+1} \in I_j^m$) and let I^\star be the interval which is to the right of I_j^m on the circle;*
- *merge I_j^m and I^\star and relabel the blocks as $1, \dots, n - m - 1$ in arbitrary order.*

Chassaing & Louchard remarked a connection between their parking scheme with Pavlov forests. Let us briefly explain this connection. For $1 \leq m \leq n - 1$ and $1 \leq i \leq n$, let

$$\mathcal{C}_i^m = \{j : C_j = i, 1 \leq j \leq m\} \quad (14)$$

be the set of cars among the first m ones that have chosen the place i (and then parked at some position after i on the circle). By construction, the distribution of $(\#\mathcal{C}_i^m, 1 \leq i \leq n)$ is multinomial: for a vector of non negative integers $(c_i, 1 \leq i \leq n)$ satisfying $\sum_{1 \leq i \leq n} c_i = m$,

$$\mathbb{P}(\#\mathcal{C}_i^m = c_i, 1 \leq i \leq n) = \binom{m}{c_1, \dots, c_n} / n^m. \quad (15)$$

Set

$$S_k^m := \sum_{i=1}^k (\#\mathcal{C}_i^m - 1), \quad \text{for } 0 \leq k \leq n, \quad (16)$$

which is a “bridge” process with exchangeable increments. This is reminiscent of the Lukasiewicz walk for Poisson Galton-Watson forests. Indeed, under the “excursion condition”

$$(\text{ExcCond}) : S_k^{n-1} \geq 0, \quad \text{for } 0 \leq k \leq n - 1 \quad \text{and} \quad S_n^{n-1} = -1, \quad (17)$$

$(S_k^{n-1}, 0 \leq k \leq n)$ has the same distribution as the Lukasiewicz walk of a Poisson Galton-Watson tree conditioned to have size n .

On the other hand, by employing a cyclic permutation (see Remark 16 below for more details), we have the following.

Proposition 14 (Chassaing & Louchard [13]). *(13) also holds true conditionally on (ExcCond).*

A bijection can be established between a tree and the state of the parking at time $n - 1$ as follows. The root of the tree is a node labelled 0. For $1 \leq i \leq n$, \mathcal{C}_i^{n-1} is the label set of the children of the i th node (in depth-first order); so in particular, the i th node has $\#\mathcal{C}_i^{n-1}$ children. See Fig. 3 for an example. In a similar way, when only m cars have parked, the sequence $(C_i, 1 \leq i \leq m)$ encodes a Pavlov forest with $n - m$ components (which is a forest of $n - m$ trees whose roots are labelled by r_1, \dots, r_{n-m} and whose non-root nodes are labelled by $[m]$), $1 \leq m \leq n - 1$ (see also Fig. 3).

Even if the combinatorial correspondence explained above is exact, one may feel it is not really satisfactory in the sense that in the additive coalescent, what we add as time goes by are edges not nodes! For this reason, one might expect (C_i) to encode edges rather than nodes.

In what follows we explain how to implement this combinatorial encoding.

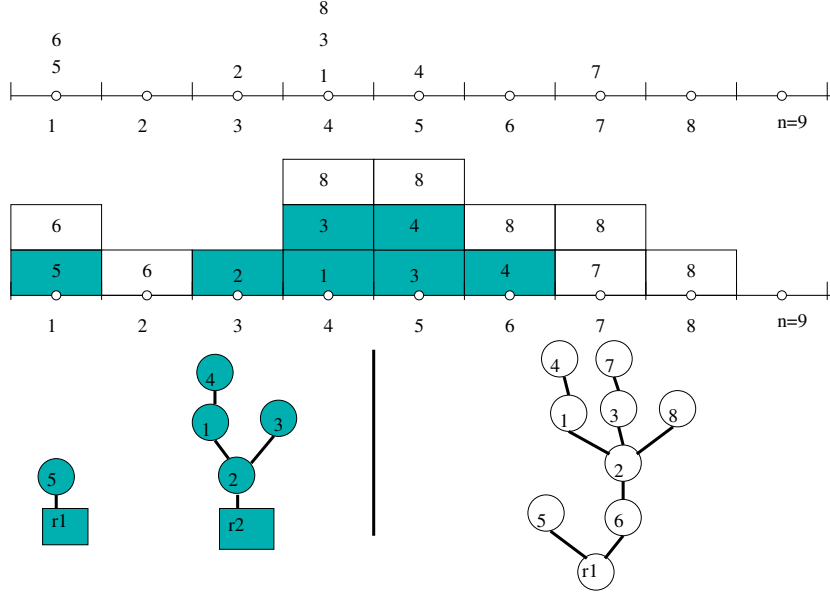


Figure 3: An example of Chassaing & Louchard's construction. In the first picture, the numbers above the interval i , from bottom to top, are the successive labels of the cars which have chosen Place i upon arrival (e.g. the cars 1, 3 and 8 have all chosen Place 4). The middle one is an illustration of the different tries of each car before finding a free place. The blue cases correspond to the state of the system when the first 5 cars have parked. The corresponding Pavlov forest is depicted in blue on the left in the last line, while the right tree is the final tree associated with the parking scheme in this example.

2.6 A new connection via an enriched parking process

The number of different parking histories is given by the number of vectors $(C_i, 1 \leq i \leq n)$, which is n^{n-1} . A simple counting argument shows that standard parkings can not faithfully encode the process $(F_k^{\text{PF}}, 1 \leq k \leq n-1)$, nor $((F_k^{\text{SC}}, L_k^{\text{SC}}), 1 \leq k \leq n-1)$. Indeed, the number of different forest histories is:

- $n^{n-1}(n-1)!$ for $(F_k^{\text{PF}}, 1 \leq k \leq n-1)$ since this is the number of different choices for the directed added edges,
- $n!(n-1)!$ for $((F_k^{\text{SC}}, L_k^{\text{SC}}), 1 \leq k \leq n-1)$ for the same reason,
- to encode the additive coalescent, the pairs of blocks that merge at each moment need to be encoded, so that the number of histories is $n!(n-1)!(2^{n-1})$.

Therefore, we need to enrich the parking structure in order to encode the forest processes or the coalescent. We opt for an encoding which carries an extra structure bringing therefore an additional factor $(n-1)!$ into enumeration compared to the usual parking scheme. This extra structure is needed to encode Pitman's forests process. Actually, it allows us to encode not only the $((F_k^{\text{SC}}, L_k^{\text{SC}}), 1 \leq k \leq n-1)$ but also the random environment where the sequence is built. To be more precise, we will consider some enriched parking schemes which bring an extra factor $n!$; the superfluous factor n is then taken care of by considering the rotation equivalence classes of the parkings (see Remark 16).

2.6.1 Encoding of additive coalescent by an enriched parking process

To produce an enriched parking process, we take here as data a pair $(\sigma, (C_i, 1 \leq i \leq n-1))$ where as previously $(C_i, 1 \leq i \leq n-1)$ is a vector taken in $[n]^{n-1}$ and σ is a permutation taken in $\mathcal{S}[n]$, the symmetric group over $[n]$.

Enriched parking algorithm.

- Run the **parking algorithm** defined in Section 2.5 using the sequence $(C_i, 1 \leq i \leq n-1)$ and obtain a sequence $(I[m], 0 \leq m \leq n-1)$ of partitions of $[n]$.
- For $0 \leq m \leq n-1$, define

$$\text{CC}_j^m := \{\sigma_i, i \in I_j^m\} \quad \text{and} \quad \text{CC}[m] = \{\text{CC}_j^m, 0 \leq j \leq n-m\}, \quad (18)$$

the *connected components at time m* . For each m , $\text{CC}[m]$ is a partition of $[n]$.

In the enriched parking, $I[m]$ encodes the connected components but is not itself a collection of connected components, this role being played by $\text{CC}[m]$: the parking places are still labelled from 1 to n , but the i th place of the parking corresponds to the element/node σ_i . For those CC_j^m defined in (18), we will sometimes say that CC_j^m is above I_j^m .

In the sequel we will consider the case in which the C_1, \dots, C_{n-1} are i.i.d. random variables uniformly distributed in $[n]$, and that $\sigma \sim \text{Uniform}(\mathcal{S}[n])$ and is independent from the C_i 's. In particular, this will ensure the exchangeability of the parking blocks in our construction.

Note 15. *As in Note 11, it is convenient to consider that $(\sigma, (C_1, \dots, C_{n-1}))$ are defined on a probability space $(\Omega^{\text{EPar}}, \mathcal{A}^{\text{EPar}}, \mathbb{P}^{\text{EPar}})$ where $\Omega^{\text{EPar}} = [n]^{n-1} \times \mathcal{S}[n]$, $\mathcal{A}^{\text{EPar}} = \text{Powerset}(\Omega^{\text{EPar}})$, and \mathbb{P}^{EPar} is the uniform distribution on Ω^{EPar} . On this space, the functions C_i are defined by $C_i(\omega \times \omega') = \pi_i(\omega)$ (the extraction of the i th coordinate “of the $[n]^{n-1}$ component”), $\sigma(\omega \times \omega') = \omega'$. Hence, the C_i 's and σ are random variables on this space, and the other quantities defined in the algorithms (for example $\text{CC}[m]$) are also random variables defined on this probability space.*

Remark 16. [Equivalence classes of enriched parkings] *Let us say that two pairs of data $(\sigma, (C_i, 1 \leq i \leq n-1))$ and $(\tilde{\sigma}, (\tilde{C}_i, 1 \leq i \leq n-1))$ are equivalent if there exists some $k \in [n]$ such that*

$$\begin{cases} \tilde{\sigma}(\overline{k+j}) = \sigma(j), & \text{for any } j \in [n], \\ \overline{k+C_j} = \tilde{C}_j, & \text{for all } 1 \leq j \leq n-1. \end{cases} \quad (19)$$

Then each equivalence class contains exactly n elements. To distinguish, we add a \sim to the variables $\text{CC}[m]$, C_i^m and S_i^m which are defined using $(\tilde{\sigma}, (\tilde{C}_j, 1 \leq j \leq n-1))$. It is clear that

$$\text{CC}[m] = \widetilde{\text{CC}}[m], \quad \text{for } 0 \leq m \leq n-1.$$

Namely, equivalent data produces same connected component processes, since their enriched parking processes are equal up to a rotation. Recall from (16) the definition of S^{n-1} . Moreover, by taking $k = \min\{j \in \{1, \dots, n\} : S_j^{n-1} = \min S^{n-1}\}$, we find exactly one pair of $(\sigma, (C_j, 1 \leq j \leq n-1))$ in

each equivalence class for which (ExcCond) is obeyed. This trick is classical and sometimes referred to as the rotation principle or the cyclic lemma in the literature (see Dvoretzky & Motzkin [15]). In particular, conditioning on (ExcCond) does not change the distribution of $(\text{CC}[m], 0 \leq m \leq n-1)$ by the previous argument (a parking under condition (ExcCond) is called “confined parking sequence” in [13].)

Notice that the block sizes $(|I[m]|, 0 \leq m \leq n-1)$ are the same for the parking algorithm and for the enriched version, so that Propositions 12 and 14, as well as the representation provided by Remark 13 hold for the enriched version. In contrast to $|I[m]|$, which are just some interval lengths, $\text{CC}[m]$ are partitions, with exchangeable elements. As a consequence, we have the following, which implies both Propositions 12 and 14.

Proposition 17. *The following distributional equality holds*

$$(\text{CC}[m], 0 \leq m \leq n-1) \stackrel{(d)}{=} (\Pi_m^+, 0 \leq m \leq n-1). \quad (20)$$

Further, (20) also holds conditionally on (ExcCond).

Proof. For $m \in [n-2]$, let us show that conditional on $(\text{CC}[j], j \leq m)$, two blocks CC_j^m and CC_i^m of $\text{CC}[m]$ merge at time m with probability $(|\text{CC}_j^m| + |\text{CC}_i^m|)/(n(n-m-1))$. By the definition of Π^+ , this will entail (20). Intuitively, the proof is the same as that of Proposition 12 since it relies on an exchangeability property in the parking. In [13], a counting argument is provided. Here, we will instead detail a symmetry argument.

First, notice that the sequence of connected components $(\text{CC}_j^m, 1 \leq j \leq n-m)$ can be equipped with a total order inherited from the order between their minimal elements: $\text{CC}_j^m < \text{CC}_i^m \Leftrightarrow \min(\text{CC}_j^m) < \min(\text{CC}_i^m)$. Up to a relabelling of the $\text{CC}[m]$, we may and will assume that the index j in $(\text{CC}_j^m, 1 \leq j \leq n-m)$ corresponds to the rank for this order. Recall that the CC_j^m 's correspond to some intervals I_j^m 's (we said the CC_j^m were above them).

For any s_1, \dots, s_k positive integers summing to n , let $\mathbf{X}[s_1, \dots, s_k]$ be the set of k -tuples (I_1, \dots, I_k) of intervals that forms a partition of $\overline{[1, n]}$, and such that $|I_j| = s_j$ for any j . The formula,

$$\#\mathbf{X}[s_1, \dots, s_k] = nk!/k = n(k-1)! \quad (21)$$

comes from the fact that given the sizes of the intervals, their respective order around the circle is given by a permutation considered up to a rotation (this provides the factor $(k-1)!$), and then their position along the circle is specified by fixing the position of the first one, which gives the additional factor n .

Now, we claim that the distribution of $I[m] := (I_j^m, 1 \leq j \leq m)$ conditionally on $\text{CC}[m]$ is uniform in $\mathbf{X}[|\text{CC}_1^m|, \dots, |\text{CC}_{n-m}^m|]$: for any element $I_{[n-m]}^* := (I_j^*, 1 \leq j \leq n-m)$ in this set

$$\mathbb{P}\left(I[m] = I_{[n-m]}^* \mid \text{CC}[m]\right) = \frac{1}{n(n-m-1)!}. \quad (22)$$

This can be proved by computing the number of pairs $(\sigma, (C_j, 1 \leq j \leq m))$ for which the partitions above a sequence of intervals $I_{[n-m]}^* \in \mathbf{X}[|\mathbf{CC}_1^m|, \dots, |\mathbf{CC}_{n-m}^m|]$ are exactly the \mathbf{CC}_j^m 's, and to see that this number is the same for any $I_{[n-m]}^*$ in this set.

But since $(\sigma, (C_j, 1 \leq j \leq m))$ is uniformly distributed on $\mathcal{S}[n] \times [n]^m$, to prove (22), we can just show that the number of pairs $(\sigma, (C_j, 1 \leq j \leq m)) \in \mathcal{S}[n] \times [n]^m$ which produce $\mathbf{CC}[m]$ above $I_{[n-m]}^*$ is the same for any $I_{[n-m]}^* \in \mathbf{X}[|\mathbf{CC}_1^m|, \dots, |\mathbf{CC}_{n-m}^m|]$. A symmetry argument suffices:
– first, σ and $(C_j, 1 \leq j \leq m)$ are both invariant by translation:

$$((\sigma_i, 1 \leq i \leq n), (C_i, 1 \leq i \leq n)) \stackrel{(d)}{=} ((1 + \sigma_i, 1 \leq i \leq n), (1 + C_i, 1 \leq i \leq n)).$$

– second, the probability of an interval configuration is unchanged if we permute two adjacent intervals: consider two consecutive intervals $I = [\overline{a, b-1}]$, $I' = [\overline{b, c}]$ and the map f defined by $f(x) = x$ if $x \notin I \cup I'$, $f(x) = \overline{a + x - b}$ if $x \in I'$, $f(x) = \overline{x + c - b}$ if $x \in I$, then again

$$((\sigma_i, 1 \leq i \leq n), (C_i, 1 \leq i \leq n)) \stackrel{(d)}{=} ((\sigma_{f(i)}, 1 \leq i \leq n), (C_{f(i)}, 1 \leq i \leq n)).$$

These two properties ensure that the intervals below the $\mathbf{CC}[m]$ are exchangeable, and invariant by translation.

Next, in the enriched parking algorithm, two blocks \mathbf{CC}_j^m and \mathbf{CC}_i^n will be merged if the corresponding intervals I_i^m and I_j^m are neighbours and if C_{m+1} falls in the first one (around the circle). Since (22) implies that I_j^m is next to I_i^m with probability $1/(n-m-1)$. Taking into account that I_j^m can be to the left or to the right of I_i^m , we find that C_{m+1} will produce a merging of I_j^m and I_i^m with probability $(|I_j^m| + |I_i^m|)/(n(n-m-1))$, that is, proportional to $(|I_j^m| + |I_i^m|)$. Finally, (20) also holds conditionally on (ExcCond), since we have seen in Remark 16 that the conditioning does not change the distribution of $(\mathbf{CC}[m], 0 \leq m \leq n-1)$. \square

Now, we introduce two constructions of graphs where $\mathbf{CC}[m]$ will appear as vertex sets of the connected components there. In consequence, we obtain, on the probability space Ω^{EPar} where $(C_i, 1 \leq i \leq n-1)$ and σ are defined, a coupling between the enriched parking process, Pitman's forests $(F_k^{\text{PF}}, 0 \leq k \leq n-1)$ and the forests with decreasing labelling $((F_k^{\text{SC}}, L_k^{\text{SC}}), 0 \leq k \leq n-1)$.

2.6.2 Encoding of Pitman forest

For $0 \leq m \leq n-1$, we associate with the partition $\mathbf{CC}[m]$ defined in (18) a forest belonging to $\mathcal{F}_{n-m,n}$, whose tree components induce the same partition of $[n]$ as $\mathbf{CC}[m]$. We then show that the sequence of the forests is distributed as $(F_k^{\text{PF}}, 0 \leq k \leq n-1)$.

Recall that $\sigma \sim \text{Uniform}(\mathcal{S}[n])$ and $C_i, 1 \leq i \leq n-1$, are i.i.d. uniformly distributed in $[n]$. For a circular interval I , set $L(I)$ to be its leftmost element of I , which is the element $\ell \in I$ such that $\ell-1 \notin I$. In the beginning, the intervals I_j^0 's are the trivial intervals $[j, j]$ for $1 \leq j \leq n$.

Algo 1.

For m varying from 0 to $n - 2$, do the following:

- Let $j \in \{1, \dots, n - m\}$ be the index such that $C_{m+1} \in I_j^m$ and let I^* be the next one on the circle.
- Add an edge from $\sigma_{L(I^*)}$ to $\sigma_{C_{m+1}}$ and label it as $m + 1$ (see the second line in Fig. 4 for an example).
- Relabel the intervals as $1, \dots, n - m - 1$.

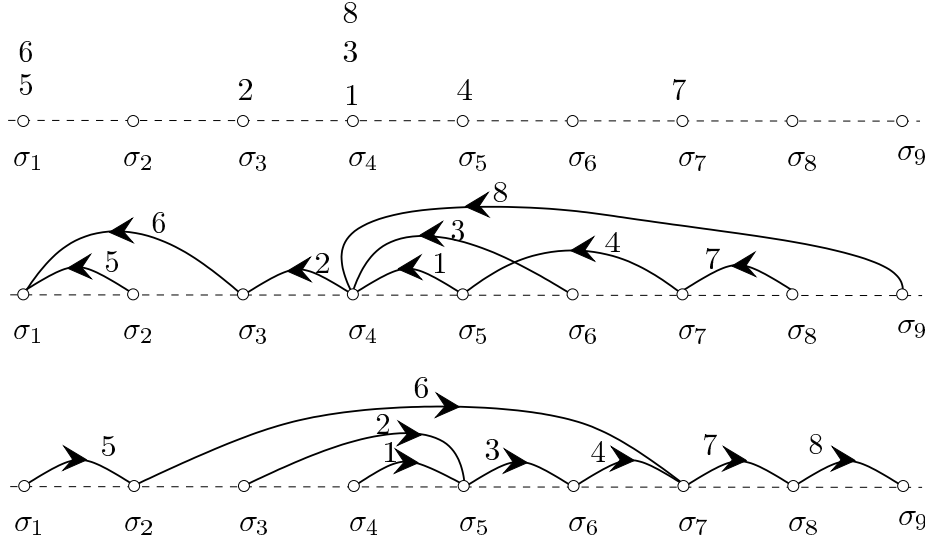


Figure 4: In the first line, the numbers above σ_k are the labels of cars which have chosen Place k . In this example, $C_1 = C_3 = C_8 = 4, C_2 = 3, C_4 = 5, C_5 = C_6 = 1, C_7 = 7$. In the second line, an illustration of the construction of Pitman's forest: add an edge between $\sigma_{C_{m+1}}$ and the left endpoint of the next interval. Turn the picture clockwise by $\pi/2$ degrees and obtain the final tree of Pitman's forest. Discarding the edges with labels $> m$ gives the forest after m coalescences. In the third line, the illustration for the construction of T^{SC} : add an edge between the right endpoint of the connected component containing $\sigma_{C_{m+1}}$ and the right endpoint of the next interval. Turn the picture clockwise by $\pi/2$ degrees and obtain the canonical planar embedding of a rooted tree equipped with decreasing edge-labels. Discarding the edges with labels $> m$ gives the corresponding forest after m coalescences. The left-depth first traversal of the tree is given by $(\sigma_9, \sigma_8, \dots, \sigma_1)$. Observe that the connected components of both constructions induce the same partitions of the nodes.

It is easy to see that **Algo 1** produces an increasing sequence of forests, since the edges go from right to the left and at each position there is at most one outgoing edge.

Proposition 18. *Algo 1 produces a sequence of labelled forests which has the same distribution as $(F_k^{\text{PF}}, 0 \leq k \leq n - 1)$ introduced just above Theorem 2. Denote this sequence of forests also by $(F_k^{\text{PF}}, 0 \leq k \leq n - 1)$. If we write $\text{CC}(F_k^{\text{PF}})$ for the partitions induced by the connected components of F_k^{PF} , we get $(\text{CC}(F_k^{\text{PF}}), 0 \leq k \leq n - 1) = (\text{CC}[k], 0 \leq k \leq n - 1)$.*

Proof. This is an immediate consequence of the fact that **Algo 1** encodes exactly Pitman construction of $(F_k^{\text{PF}}, 0 \leq k \leq n-1)$ using a linear parking. To figure out this fact, observe Fig. 4. At time 0, no edge is present, each node σ_j constitutes a block. Edge additions produce block merging, but at any time m , each block b corresponds to a tree t , and the leftmost node u of b is the root of t . Step 2 of **Algo 1** conserves this property throughout the construction. Now let us work only on the set of trees $f_m = \{t_1, \dots, t_{n-m}\}$ present at time m , and consider an edge addition. The node $\sigma_{C_{m+1}}$ is uniform on the set of nodes. It corresponds then to the choice of the uniform node α_{m+1}^{PF} in the Pitman forest. Now, (22) in the proof of Proposition 17 implies that $\sigma_{L(I^*)}$, which is the tail of the edge to be added, is distributed as the root of a uniform tree in the remaining forest, namely, it has the same distribution as β_{m+1}^{PF} . \square

2.6.3 Encoding of the forests with decreasing labellings

Recall that $(\text{CC}[k], 0 \leq k \leq n-1)$ has been defined in Section 2.6.1 using $\sigma \sim \text{Uniform}(\mathcal{S}[n])$ and $C_i, 1 \leq i \leq n-1$, i.i.d. and uniformly distributed in $[n]$. We construct a sequence of forests such that $\text{CC}[k]$ appear as the vertex sets of the connected components in the k th forest. And we will see that the sequence coincides in distribution with $((F_k^{\text{SC}}, L_k^{\text{SC}}), 0 \leq k \leq n-1)$ as defined in Section 2.1.

For a circular interval I , let $R(I)$ be its rightmost element, i.e. the element $r \in I$ such that $r+1 \notin I$. In the beginning, the intervals I_j^0 are the trivial intervals $[j, j]$ for $1 \leq j \leq n$.

Algo 2.

For m varying from 0 to $n-2$ do the following:

- Let $j \in \{1, \dots, n-m\}$ be the index such that $C_{m+1} \in I_j^m$, and let I^* be the next one around the circle.*
- Add an edge from $\sigma_{R(I_j^m)}$ to $\sigma_{R(I^*)}$ and label the edge as $m+1$.*
- Relabel the intervals as $1, \dots, n-m-1$.*

Again, it is easy to see that **Algo 2** produces an increasing sequence of forests, since the edges are directed from left to the right and at each position there is at most one outgoing edge. Furthermore, since edges are always added between the current roots, the forests constructed in this way are all equipped with a decreasing edge-labelling.

Proposition 19. *Algo 2 produces a sequence of labelled forests which has the same distribution as $((F_k^{\text{SC}}, L_k^{\text{SC}}), 0 \leq k \leq n-1)$ described in Section 2.1. Denote this sequence of forests also by $((F_k^{\text{SC}}, L_k^{\text{SC}}), 0 \leq k \leq n-1)$. If we write $\text{CC}(F_k^{\text{SC}})$ for the partitions induced by the connected components of F_k^{SC} , we get $(\text{CC}(F_k^{\text{SC}}), 0 \leq k \leq n-1) = (\text{CC}[k], 1 \leq k \leq n-1)$.*

Proof. Again this is a consequence of the fact that **Algo 2** is just a linear encoding of the construction of $(F_k^{\text{SC}}, 0 \leq k \leq n-1)$. As in the proof of Proposition 18, observe Fig. 4. Throughout the construction, each block I in the parking corresponds to a tree. The set $\{\sigma_i, i \in I\}$ coincides with the vertex sets of the corresponding tree, and moreover, $\sigma_{R(I)}$, “the right endpoint of each block” is the root of the corresponding tree (this property is guarantied by the second point of **Algo 2**). The rest of the proof can be easily adapted from that of Proposition 18. \square

2.7 Trees with decreasing labellings and canonical orderings

We may define several orders on a Cayley tree equipped with a decreasing edge-labelling (T, L) . Using the edge labelling, we may order the child-edge of each node. This induces an order on the nodes, that we call, canonical order. Using this order, we can “draw” a tree in the plane, drawing the in-going edges of each node in such a way that the edge labels are increasing from left to right. Call this the canonical embedding of T and denote it by $\text{Can}(T, L)$. Then the depth-first search of $\text{Can}(T, L)$ defines a linear order on the nodes of T which we call the *left-depth first traversal* of T . **Algo 2** produces a decreasing tree with a given left-depth first traversal. Indeed, using the fact that in **Algo 2** edges are always added between the right endpoints of two neighboring intervals, one can readily check that the left-depth first traversal of the tree T^{SC} defined by **Algo 2** is given by $(\sigma_{\overline{j_0+1-j}}, 1 \leq j \leq n)$, where σ_{j_0} is the root of T^{SC} (see the last line in Fig. 4). This leads to the following.

Proposition 20. *Let $T \in \mathcal{F}_{1,n}$ and $L \in \text{DL}(T)$. The number of elements $(\sigma, (C_j, 1 \leq j \leq n-1)) \in \Omega^{\text{Epar}}$ (see Note 15) which produces (T, L) in **Algo 2** is given by $n \prod_{u \in T \setminus \{r(T)\}} |\text{Sub}_u(T)|$.*

Notice that the arguments in Remark 16 also entail that elements in the same rotation class produce the same sequence $((F_k^{\text{SC}}, L_k^{\text{SC}}), 0 \leq k \leq n-1)$ in **Algo 2**. Thus, the number of rotation classes that are sent onto (T, L) by **Algo 2** is $\prod_{u \in T \setminus \{r(T)\}} |\text{Sub}_u(T)|$. This is consistent with Lemma 5 which says that $\#\text{DL}(T) = (n-1)! / \prod_{u \in T \setminus \{r(T)\}} |\text{Sub}_u(T)|$ so that the total number of rotation classes is indeed $(n-1)!n^{n-1}$ as expected.

Proof. Since elements in the same rotation class produce the same $(T^{\text{SC}}, L^{\text{SC}})$ by **Algo 2**, consider the unique element $(\sigma, (C_i, 1 \leq i \leq n))$ of the rotation class such that the root of T^{SC} is σ_n . In this case, **Algo 2** outputs (T, L) from $(\sigma, (C_i, 1 \leq i \leq n))$ if and only if $\sigma_{C_j} \in \text{Sub}_{u_j}(T)$ where u_j is the node from which the edge labelled j is directed, for $1 \leq j \leq n-1$. As each non-root node of T appears once in $(u_j, 1 \leq j \leq n-1)$, we easily conclude. \square

We give some further explanation in the link between $(T^{\text{SC}}, L^{\text{SC}})$ and $(\sigma, (C_j, 1 \leq j \leq n))$, in describing precisely where the $(m+1)$ th edge is added. Recall that $\mathcal{C}_j^m = \{j : C_j = i, 1 \leq j \leq m\}$ and that $S_k^m = \sum_{i=1}^k (\#\mathcal{C}_i^m - 1)$. We further suppose **(ExcCond)**. We recognize $(S_k^m, 0 \leq k \leq n)$ as the Lukasiewicz walk for Pitman’s forest F_m^{PF} . Under the condition **(ExcCond)**, the partition $I[m]$, which encodes the connected components of F_m^{PF} , corresponds to the excursion intervals of $S^m - \underline{S}^m$, where $\underline{S}_j^m := \min\{S_i^m : i \leq j\}$. But F_m^{PF} and F_m^{SC} share the same connected components by construction. It follows that the rightmost node of the block which contains C_{m+1} is $\sigma_{R_{m+1}}$ where

$$R_{m+1} := \min\{j \geq C_{m+1} : S_j^m = \underline{S}_j^m\}.$$

Similarly, the rightmost node of the next block is given by $\sigma_{R'_{m+1}}$, where

$$R'_{m+1} := \min\{j \geq R_{m+1} + 1 : S_j^m = \underline{S}_j^m\}.$$

Then the edge labelled $m+1$ is from $\sigma_{R_{m+1}}$ to $\sigma_{R'_{m+1}}$.

2.8 Coupling between $(F_k^{\text{PF}}, 0 \leq k \leq n-1)$ and $((F_k^{\text{SC}}, L_k^{\text{SC}}, 0 \leq k \leq n-1)$

Everything has already been said on this point. We collect in this section some simple facts spread throughout the paper.

In Sections 1.1 and 1.2, the two processes $F^{\text{PF}}[n-1] = (F_k^{\text{PF}}, 0 \leq k \leq n-1)$ and $FL^{\text{SC}}[n-1] = ((F_k^{\text{SC}}, L_k^{\text{SC}}), 0 \leq k \leq n-1)$ have been defined using respectively $(\alpha_k^{\text{PF}}, \beta_k^{\text{PF}})$ and $(\alpha_k^{\text{SC}}, \beta_k^{\text{SC}})$. In Sections 2.6.2 and 2.6.3, relying on the enriched parking scheme and using as data $(\sigma, (C_i)_{1 \leq i \leq n-1}) \sim \text{Uniform}(\mathcal{S}[n] \times [n]^{n-1})$, we obtain an alternative construction of $F^{\text{PF}}[n-1]$ (resp. of $FL^{\text{SC}}[n-1]$). It follows that both $F^{\text{PF}}[n-1]$ and $FL^{\text{SC}}[n-1]$ are defined on the same probability space where $(\sigma, (C_i)_{1 \leq i \leq n-1})$ is defined. Observe that this coupling between $F^{\text{PF}}[n-1]$ and $FL^{\text{SC}}[n-1]$ amounts to

- taking $\alpha_k^{\text{SC}} = \alpha_k^{\text{PF}}$,
- taking β_k^{SC} to be the root of the tree in F_k^{SC} which contains the node with label β_k^{PF} .

We assume from now on that $F^{\text{PF}}[n-1]$ and $FL^{\text{SC}}[n-1]$ are coupled in this way. Observe that at each instant, the partitions induced by the two forests coincide, and new edges are added between the same pairs of connected components.

The information contained in $FL^{\text{SC}}[n-1]$ is not sufficient to reconstitute the rotation class of $(\sigma, (C_j, 1 \leq j \leq n-1))$. The reason is that since we add the edge between the roots of trees, then the α_k^{SC} 's (which corresponds to the C_i 's) can not be recovered from $FL^{\text{SC}}[n-1]$. Notice that $F^{\text{PF}}[n-1]$ is a sequence of forests without edge-labelling. On the other hand, since the difference between F_k^{PF} and F_{k-1}^{PF} is the added edge at time k , this process allows one to recover the sequence $((\alpha_k^{\text{SC}}, \beta_k^{\text{SC}}), 1 \leq k \leq n-1)$. Hence, $F^{\text{PF}}[n-1]$ characterizes $FL^{\text{SC}}[n-1]$, but the converse is false.

Nevertheless, the distribution of $F^{\text{PF}}[n-1]$ knowing $FL^{\text{SC}}[n-1]$ can be described: assume that we have been able to construct a sequence of forests $(F_k^{\text{PF}}, 0 \leq k \leq m)$ thanks to $((F_k^{\text{SC}}, L_k^{\text{SC}}), 0 \leq k \leq m)$ for some m , such that both forests have the expected distribution. Assume moreover that the partitions induced on $[n]$ by these forests coincide at time m (at time 0, the property is clear). Now, observe in $(F_{m+1}^{\text{SC}}, L_{m+1}^{\text{SC}})$ the edge labelled $m+1$. It goes from the root u_k of a tree to the root v_k of another tree. By construction, we know that $u_k = r(\alpha_k^{\text{SC}})$, and $v_k = \beta_k^{\text{SC}}$. Consider in F_m^{PF} the tree t_k which contains u_k , and t'_k which contains v_k (recall that the nodes are labelled). To construct F_{m+1}^{PF} , add an edge from the root of t'_k to the node w_k chosen uniformly in t_k .

2.9 Asymptotics

The asymptotic of additive coalescent processes is known to be described by the Brownian excursions as follows. Let $\mathbf{e} = (\mathbf{e}(x), 0 \leq x \leq 1)$ be the normalized Brownian excursion of length 1, and consider the operator

$$\begin{aligned} \Psi : C[0,1] &\longrightarrow C[0,1] \\ f &\longmapsto \Psi(f) : \begin{aligned} [0,1] &\longrightarrow \mathbb{R}^+ \\ x &\longmapsto f(x) - \min\{f(y), y \leq x\} \end{aligned} \end{aligned} \quad (23)$$

For any $\lambda \geq 0$, let \mathbf{e}_λ be the process

$$\mathbf{e}_\lambda(x) = \mathbf{e}(x) - \lambda x, \quad x \in [0,1]. \quad (24)$$

Let $f \in C[0,1]$. An interval $I = [a, b]$ with $b > a$ is said to be an excursion of f above its running minimum if $b = \inf\{t > a, f(t) = f(a)\}$ and if $f(a) = f(b) = \min\{f(t), t \leq b\}$.

Denote by $\gamma^+(\lambda)$ the sequence of excursion lengths of $\Psi(\mathbf{e}_\lambda)$ above its running minimum which are sorted in decreasing order so that $\gamma^+(\lambda) \in \ell_\downarrow^1$, as defined in (5). Consider for each n and for any $\lambda > 0$, the sequence $\gamma^n(\lambda)$ defined as the rearrangement of $(|\text{CC}_j^{[n-\lambda\sqrt{n}]}|/n, 1 \leq j \leq \lfloor \lambda\sqrt{n} \rfloor)$ in decreasing order. This finite sequence completed by a sequence of zeros, is considered as an element of ℓ_\downarrow^1 .

Theorem 21. *The following convergence holds in $\mathbb{D}([0, +\infty), \ell_\downarrow^1)$*

$$(\gamma^n(\lambda), \lambda \geq 0) \xrightarrow[n \rightarrow \infty]{(d)} (\gamma^+(\lambda), \lambda \geq 0). \quad (25)$$

Several proofs of this theorem exist in the literature. Evans & Pitman [16] showed that the additive coalescent process is a Feller process. Aldous & Pitman [5] gave a description of the fragmentation associated with the additive coalescent by studying the fragment sizes of a cutting process on the Brownian CRT (however, no connection was made yet with \mathbf{e}_λ). The above representation of the standard additive coalescent in terms of the excursion lengths of $(\mathbf{e}_\lambda, \lambda \geq 0)$ first appear in two almost simultaneous papers : Bertoin [7] who discusses only the continuous case, and Chassaing & Louchard [13] paper about parking scheme who prove the convergence of the finite dimensional distribution.

In Broutin & Marckert [11], the edges $(e \in E(T))$ of a uniform Cayley tree T are labelled by some i.i.d. uniform random variables $U[T] = (U_e, e \in E(T))$ on $[0, 1]$. Denote the forest $F_n(t)$ obtained by discarding the edges with labels greater than t . The process $(F_n(t), t \in [0, 1])$ coincides with the additive coalescent, up to a time change. In [11], it is explained that the Prim order on the tree T equipped with $U[T]$ can be used to define a traversal of T , and further a parking process. The block sizes in the parking are then a faithful encoding of the tree sizes in the forest process $F_n(t)$, and the coalescence coincides with the coalescence of adjacent parking blocks.

In Propositions 18 and 19, we observed that each of **PF Algo** and **SC Algo** provides a coupling between a parking process and some process of coalescing forests. Each of the three processes encodes the additive coalescent. However, our present construction has the advantage that the parking/forest constructions do not need a pre-labelling of the edges of T , leading to an easier handling when dealing with then asymptotic behaviour. For the sake of completeness, as well as to illustrate the advantage obtained in terms of argument concision, we provide here a short proof of Theorem 21.

Recall the random walk $(S_k^{n-1}, 0 \leq k \leq n)$ defined in (16), conditional on (ExcCond).

Remark 22. *Contrary to many other places in the paper, here the assumption of condition (ExcCond) is essential. Under this condition, the excursion lengths of S^m above its running minimum coincide with the connected component sizes of $\text{CC}[n-1]$; without the condition, this will be false because of the first and last excursions.*

First, recall the following classical result due to Kaigh [20]. Conditional on (ExcCond):

$$\left(\frac{1}{\sqrt{n}} S_{nt}^{n-1} \right)_{t \in [0, 1]} \xrightarrow[n \rightarrow \infty]{(d)} \mathbf{e} \quad \text{in } \mathbb{C}([0, 1], \mathbb{R}), \quad (26)$$

where S^{n-1} , as well as the other processes S^m that will be considered below, is interpolated between integer points.

Recall that $\mathcal{C}_j^m = \{i : C_i = j, 1 \leq i \leq m\}$. Clearly, $\#\mathcal{C}_j^m$ is non decreasing in m for any j . Let

$$D_j^m = \sum_{i=1}^j (\#\mathcal{C}_i^{n-1} - \#\mathcal{C}_i^m), \text{ for any } j \in \{0, \dots, n\},$$

which counts the number of cars which have chosen a place left to j after time m . Then,

$$S_j^m = S_j^{n-1} - D_j^m \text{ for any } j \in \{0, \dots, n\}. \quad (27)$$

Notice that D_j^m , conditional on $\{\mathcal{C}_i^{n-1}, 1 \leq i \leq n\}$ is hypergeometrically distributed :

$$\mathbb{P}(D_j^m = k \mid \#\mathcal{C}_i^{n-1}, 1 \leq i \leq n) = \frac{\binom{j+S_j^{n-1}}{k} \binom{n-1-j-S_j^{n-1}}{n-m-k}}{\binom{n-1}{n-m}}. \quad (28)$$

The formula comes from the fact that $j+S_j^{n-1} = \sum_{i=1}^j \#\mathcal{C}_i^{n-1}$ and $n-1-j-S_j^{n-1} = \sum_{i=j+1}^n \#\mathcal{C}_i^{n-1}$. Set

$$s_{n,\lambda}(t) := \frac{1}{\sqrt{n}} S_{nt}^{n-\lambda\sqrt{n}}, \quad \lambda \geq 0, t \in [0, 1] \quad (29)$$

where the discrete process $(S_x^y, 0 \leq x \leq m, 0 \leq y \leq n-1)$ is obtained by interpolating S_k^m , first “in k ” between integer points, and then in m between integer points; we thus obtain a continuous process in the end.

Theorem 23. *Take some $a \in (0, \infty)$ and $\Lambda = [0, a]$. Conditionally on (ExcCond),*

$$(s_{n,\lambda}(t))_{(t,\lambda) \in [0,1] \times \Lambda} \xrightarrow[n \rightarrow \infty]{(d)} (e_\lambda(t))_{(t,\lambda) \in [0,1] \times \Lambda}, \quad (30)$$

in $C([0, 1] \times \Lambda, \mathbb{R})$.

Proof. Let us work on a probability space where the convergence in (26) holds almost surely, in other words on this space, $s_{n,0} \xrightarrow[n \rightarrow \infty]{(a.s.)} e$. We have the right to do so thanks to the Skorokhod representation theorem. We claim that, for $\lambda > 0$ fixed, in $C[0, 1]$,

$$\left(\frac{D_{nt}^{\lfloor n-\lambda\sqrt{n} \rfloor}}{\sqrt{n}} \right)_{t \in [0,1]} \xrightarrow[n \rightarrow \infty]{(d)} (\lambda t)_{t \in [0,1]}. \quad (31)$$

There are two arguments:

- First, since $t \mapsto D_{nt}^{\lfloor n-\lambda\sqrt{n} \rfloor}$ is non decreasing, the convergence in the uniform topology will follow from the pointwise convergence (this is a consequence of Dini’s theorem).
- For fixed t , the convergence (in distribution) of $D_{nt}^{\lfloor n-\lambda\sqrt{n} \rfloor}$ to λt is a combined consequence of the hypergeometric representation (28) and the tightness of S_{nt}^{n-1}/\sqrt{n} (implied by (26)).

By (26), (27) and (31), we deduce that for a fixed λ , in $C[0, 1]$,

$$(s_{n,\lambda}(t))_{t \in [0,1]} \xrightarrow[n \rightarrow \infty]{(d)} (e_\lambda(t))_{t \in [0,1]}. \quad (32)$$

On the probability space on which we are working, the limit depends only on the limit e , the a.s. limit of $s_{n,0}$. In other words, conditional to this limit, the limit of $s_{n,\lambda}$ is deterministic, so that the uni-dimensional a.s. convergence stated in (32) implies the following joint convergence: for any $0 = \lambda_0 \leq \lambda_1 \leq \dots \leq \lambda_k$ one has

$$\left[(s_{n,\lambda_j}(t))_{t \in [0,1]}, 0 \leq j \leq k \right] \xrightarrow[n \rightarrow \infty]{(a.s.)} \left[(e_\lambda(t))_{t \in [0,1]}, 0 \leq j \leq k \right]. \quad (33)$$

This time, using the continuity and monotonicity of the limit in λ , we conclude. \square

Proof of Theorem 21. The sizes of the excursions of $s_{n,\lambda}(t)$ above its running minimum do not exactly correspond to the component sizes of $(|\mathcal{CC}_j^{[n-\lambda\sqrt{n}]}|/n, 1 \leq j \leq \lambda\sqrt{n})$. A slight adjustment has to be made: we say that $I = [a, b]$ is an excursion of s_n above its minimum if $a, b \in \{0, 1/n, \dots, n/n\}$, and if $b = \inf\{t > a, s_n(t) = s_n(a) - 1/\sqrt{n}\}$. We denote by $\text{EL}_n^\downarrow(s_{n,\lambda})$ the list of excursion lengths of $s_{n,\lambda}$ above its minimum sorted in decreasing order.

Consider again the convergence stated in (30) and assume that we are still working on a probability space on which this convergence holds a.s.. For any $\lambda \geq 0$, let

$$\text{locm}(\lambda) = \{x \in [0, 1], e_\lambda \text{ has a local minimum at abscissa } x\}. \quad (34)$$

If the values of e_λ on the set of local minima are different (for $x \neq x' \in \text{locm}(\lambda)$, $e_\lambda(x) \neq e_\lambda(x')$), then one can prove using the arguments of Aldous [4, Section 2.3] that

$$(s_{n,\lambda} \xrightarrow[n \rightarrow \infty]{} e_\lambda) \Rightarrow (\text{EL}_n^\downarrow(s_{n,\lambda}) \xrightarrow[n \rightarrow \infty]{} \gamma^+(\lambda)). \quad (35)$$

This convergence is a bit weaker than the conclusion of Theorem 21. It is somehow similar to the convergence of finite dimensional distributions, even if the set of $\lambda \in [0, a]$ for which the condition stated above on the local minima, has a.s. Lebesgue measure a . As discussed in Broutin & Marckert [11, Section 7], to end the proof we may proceed as follows: denote by $F_n^k(\lambda)$ the sum of the k largest excursion sizes in $s_{n,\lambda}$ and $F^k(\lambda)$ the sum of the k largest excursion sizes of e_λ . The process $F_n^k(\lambda)$ converges simply almost everywhere to $F^k(\lambda)$, and the process $\lambda \mapsto F^k(\lambda)$ is non increasing, càdlàg, which is also the case for $\lambda \mapsto F_n^k(\lambda)$. For this class of processes, simple convergence implies convergence in $\mathbb{D}([0, 1], \mathbb{R})$, so that

$$\left((F_n^k(\lambda), k \leq K)_{\lambda \in [0, a]} \right) \xrightarrow[n \rightarrow \infty]{(d)} \left((F^k(\lambda), k \leq K)_{\lambda \in [0, a]} \right) \quad (36)$$

in $\mathbb{D}([0, a], \mathbb{R}^K)$. The property stated in (35) allows us to conclude. \square

3 Dual fragmentation processes and numbers of cuts

In this part, we are interested in the fragmentation processes defined as the time-reversals of the two forest coalescent processes: $(F_{n-1-k}^{\text{PF}}, 0 \leq k \leq n-1)$ and $(F_{n-1-k}^{\text{SC}}, 0 \leq k \leq n-1)$. Throughout this section, we assume that $(F_k^{\text{PF}}, 0 \leq k \leq n-1)$ and $(F_k^{\text{SC}}, 0 \leq k \leq n-1)$ are constructed using the coupling in Section 2.8. As pointed out by Pitman, the first fragmentation process $(F_{n-1-k}^{\text{PF}}, 0 \leq k \leq n-1)$ is a Markov chain whose evolution can be described as follows: F_{n-1}^{PF} is a uniform element of $\mathcal{F}_{1,n}$; for $1 \leq k \leq n-1$, F_{n-1-k}^{PF} is obtained from F_{n-k}^{PF} by choosing a uniform edge among the $(n-k)$ edges of F_{n-k}^{PF} and by removing it. This process also comes under the name of *random cutting problem of uniform Cayley trees*.

Put in a slightly more general setting, a random cutting on a fixed tree t is a sequence of forests obtained from t by first sampling a uniform order on its edges and then removing the edges one by one according to this order. In this terminology, each edge removal is referred to as a *cut* on the initial tree t . A classic problem there is to find out the number of cuts necessary to isolate a given vertex. In the first paper on this topic Meir & Moon [26] were interested in the case where t is sampled uniformly on $\mathcal{F}_{1,n}$. Now let us observe from Pitman's work [29] or the remarks in the previous paragraph that the sequence $(F_{n-1-k}^{\text{PF}}, 0 \leq k \leq n-1)$ is in fact a random cutting on the uniform Cayley tree $T^{\text{PF}} := F_{n-1}^{\text{PF}}$. Moreover, if we recall that Π_k^{PF} is the partition of $[n]$ induced by the vertex sets of the tree components of F_k^{PF} and denote by $\pi_k^{\text{PF}}(v)$ the element of Π_k^{PF} which contains the vertex v , for $v \in [n]$, then the number of cuts necessary to isolate v is the number of cuts which have reduced the tree components containing v during the process $(F_{n-1-k}^{\text{PF}}, 0 \leq k \leq n-1)$, namely

$$X^{\text{PF}}(v) := \#\{k \in \{1, \dots, n-1\} : \pi_{n-1-k}^{\text{PF}}(v) \neq \pi_{n-k}^{\text{PF}}(v)\}. \quad (37)$$

Nowadays, it is well-known [28, 19, 9, 1] that if we take a random vertex $V_n \sim \text{Uniform}([n])$, then

$$n^{-\frac{1}{2}} \cdot X^{\text{PF}}(V_n) \xrightarrow[n \rightarrow \infty]{(d)} R, \quad (38)$$

where R is a Rayleigh variable, i.e. $\mathbb{P}(R > r) = \exp(-r^2/2)$, for $r \in (0, \infty)$. Here, we propose a simple proof of this fact based on the process $(F_{n-1-k}^{\text{SC}}, 0 \leq k \leq n-1)$ and its coupling with $(F_{n-1-k}^{\text{PF}}, 0 \leq k \leq n-1)$ as explained in Section 2.8.

Recall from Section 2.1 the distribution of the sequence of forests $(F_k^{\text{SC}}, 0 \leq k \leq n-1)$ and the (random) decreasing edge-labelling L^{SC} which encodes the coalescent history. By Theorem 6, the time-reversal process $(F_{n-1-k}^{\text{SC}}, 0 \leq k \leq n-1)$ can be described as follows: take $T^{\text{SC}} = F_{n-1}^{\text{SC}} \sim \text{Uniform}(\mathcal{F}_{1,n})$ and a decreasing edge-labelling $L^{\text{SC}} \sim \text{Uniform}(\text{DL}(T^{\text{SC}}))$; for $0 \leq k \leq n-1$, the forest F_k^{SC} is obtained from T^{SC} by removing all the edges with labels $> k$. Consider the following simple observation. During the fragmentation process $(F_{n-1-k}^{\text{SC}}, 0 \leq k \leq n-1)$, along each branch B of T^{SC} , edges are removed in a deterministic order, that is, independently of the choice of L^{SC} : among the set of edges of B , the first removed edge is incident to the root, then the edge incident to this edge, etc. In analogy to the random cutting problem, let us define for each vertex $v \in [n]$,

$$X^{\text{SC}}(v) := \#\{k \in \{1, \dots, n-1\} : \pi_{n-1-k}^{\text{SC}}(v) \neq \pi_{n-k}^{\text{SC}}(v)\}, \quad (39)$$

where $\pi_k^{\text{SC}}(v)$ is the vertex set of the tree component of $F_k^{\text{SC}}(v)$ which contains v , $0 \leq k \leq n-1$. We have the following.

Lemma 24. *For the coalescent forests $(F_{n-1-k}^{\text{SC}}, 0 \leq k \leq n-1)$ defined by the **SC Algo** and $X^{\text{SC}}(v)$ defined in (39), we have for all $v \in [n]$,*

$$X^{\text{SC}}(v) = H^{\text{SC}}(v) + D^{\text{SC}}(v), \quad (40)$$

where $H^{\text{SC}}(v)$ denotes the height of v in T^{SC} , namely, the number of edges on the unique path of T^{SC} from the root to v , and $D^{\text{SC}}(v)$ denotes the in-degree of v in T^{SC} . Moreover, under the coupling in Section 2.8, we have

$$X^{\text{PF}}(v) = X^{\text{SC}}(v), \quad \text{for all } v \in [n]. \quad (41)$$

Proof. For the first statement, observe that in the sequence $(F_{n-1-k}^{\text{SC}}, 0 \leq k \leq n-1)$, the vertex v is isolated after each edge on the path between the root and v is removed (in the order from the root towards v) and also all the in-going edges of v are removed (in the order determined by L^{SC}). This easily leads to (40). Next, in the coupling of Section 2.8, we have $(\Pi_k^{\text{PF}}, 0 \leq k \leq n-1) = (\Pi_k^{\text{SC}}, 0 \leq k \leq n-1)$. Then comparing the definitions (37) and (39), we get (41). \square

From Lemma 24, we deduce the convergence of the numbers of cuts.

Proposition 25. *Let $X^{\text{SC}}(v)$ be defined in (39), we have*

$$\max_{v \in T^{\text{SC}}} n^{-\frac{1}{2}} |X^{\text{SC}}(v) - H^{\text{SC}}(v)| \xrightarrow[n \rightarrow \infty]{\mathbb{P}} 0. \quad (42)$$

Moreover, let $X^{\text{PF}}(v)$ be as in (37) and let $V_n \sim \text{Uniform}([n])$ for each $n \geq 1$; we have

$$n^{-\frac{1}{2}} \cdot X^{\text{PF}}(V_n) \xrightarrow[n \rightarrow \infty]{(d)} \text{R}, \quad (43)$$

where R is a Rayleigh variable defined by $\mathbb{P}(\text{R} > r) = \exp(-r^2/2)$, for $r \in (0, \infty)$.

Proof. Recall from Theorem 6 that T^{SC} is distributed as a uniform Cayley tree of size n . By (40), (42) amounts to saying that

$$n^{-\frac{1}{2}} \cdot \max_{v \in T^{\text{SC}}} D^{\text{SC}}(v) \xrightarrow[n \rightarrow \infty]{\mathbb{P}} 0. \quad (44)$$

This follows from a result due to Moon [27], who shows that $(\log n / \log \log n)^{-1} \max_{v \in T^{\text{SC}}} D^{\text{SC}}(v)$ converges in probability to 1 (so actually (44) also holds with $(\log n)^{-1}$ instead of $n^{-1/2}$). Now, for the uniform vertex V_n , (42) implies that

$$n^{-\frac{1}{2}} |X^{\text{SC}}(V_n) - H^{\text{SC}}(V_n)| \xrightarrow[n \rightarrow \infty]{\mathbb{P}} 0. \quad (45)$$

On the other hand, by Theorem 6, $T^{\text{SC}} \sim \text{Uniform}(\mathcal{F}_{1,n})$. Then by Aldous [2] (see also Le Gall [22]), $H^{\text{SC}}(V_n)/\sqrt{n}$ converges in distribution to R . Therefore, $X^{\text{SC}}(V_n)/\sqrt{n}$ also converges to the Rayleigh distribution. Combined with (41) which says that $X^{\text{SC}}(V_n) \stackrel{(d)}{=} X^{\text{PF}}(V_n)$ for all n , this completes the proof of (43). \square

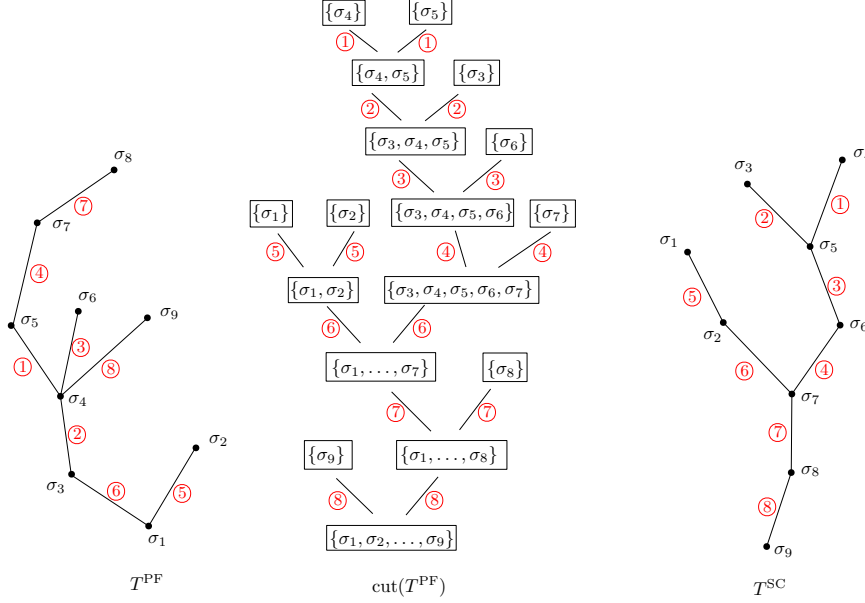


Figure 5: An illustration of T^{PF} , T^{SC} and $\text{cut}(T^{\text{PF}})$ with $n = 9$. We take the same example from Figure 4. On the left, the tree $T^{\text{PF}} = F_{n-1}^{\text{PF}}$ is obtained from **Algo 1**. The circled numbers are the labels of the edges, which correspond to the times of appearance in **Algo 1**. Removing these edges in the reverse order of these numbers gives an example of the fragmentation of T^{PF} . The tree in the middle depicts the evolution of this fragmentation in terms of the vertex set partitions. The tree on the right is the one obtained from **Algo 2**.

We end this section with a discussion on the asymptotic distribution of the pair $(T^{\text{PF}}, T^{\text{SC}})$. Let us first observe that the limit distributions for the marginals are well-known: both T^{PF} and T^{SC} are distributed as a uniform Cayley tree, thus after a rescaling $\frac{1}{\sqrt{n}}$ in the graph distance, each of them converge to Aldous' Brownian CRT [2] as $n \rightarrow \infty$. On the other hand, though we have an explicit description of the joint law from the coupling in Section 2.8, it does not ease the study for the asymptotics. However, we will show this joint law can be approximated by the law of another pair $(T^{\text{PF}}, \text{cut}(T^{\text{PF}}))$, where $\text{cut}(T^{\text{PF}})$ is the so-called cut tree of T^{PF} . The cut tree describes the genealogy of the nested partitions arising from the fragmentation process $(F_{n-1-k}^{\text{PF}}, 0 \leq k \leq n-1)$. More precisely, at step k of this process, an edge of the initial tree $T^{\text{PF}} := F_{n-1}^{\text{PF}}$ is removed, which results in that a tree is split into two subtrees and in the corresponding partition one element, say A , of Π_{n-1-k}^{PF} is split into two disjoint subsets B_1, B_2 such that $A = B_1 \cup B_2$; we then say that A has two children B_1, B_2 . This parent-child relation determines a family tree $\text{cut}(T^{\text{PF}})$ (see Fig. 5) whose vertices are the elements of $\cup_k \Pi_{n-1-k}^{\text{PF}}$. This tree has $[n]$ as its root, is binary, has $2n - 2$ edges and exactly n leaves. This is the cut tree considered by Bertoin [9].

The tree $\text{cut}(T^{\text{PF}})$ is actually intimately related to the random cutting problem in the following way. Notice that the leaves of $\text{cut}(T^{\text{PF}})$ are the singletons $\{1\}, \{2\}, \dots, \{n\}$. Write $H^{\text{cut}}(i)$ as the graph distance of $\{i\}$ in $\text{cut}(T^{\text{PF}})$ from the root. It follows from its construction (observe also

Figure. 5) that

$$H^{\text{cut}}(i) = X^{\text{PF}}(i), \quad \text{for } i = 1, 2, \dots, n. \quad (46)$$

Comparing this with Lemma 24 and Proposition 25, we have the following.

Proposition 26. *We have*

$$n^{-\frac{1}{2}} \cdot \max_{v \in [n]} |H^{\text{SC}}(v) - H^{\text{cut}}(v)| \xrightarrow[n \rightarrow \infty]{\mathbb{P}} 0; \quad (47)$$

moreover, let d^{SC} (resp. d^{cut}) denote the graph distance of T^{SC} (resp. of $\text{cut}(T^{\text{PF}})$), then

$$n^{-\frac{1}{2}} \cdot \max_{v, v' \in [n]} |d^{\text{SC}}(v, v') - d^{\text{cut}}(\{v\}, \{v'\})| \xrightarrow[n \rightarrow \infty]{\mathbb{P}} 0. \quad (48)$$

Proof. The convergence (47) is an immediate consequence of (41), (42) and (46). Let us show (48). To that end, we label the edges of $\text{cut}(T^{\text{PF}})$ in the following way: recall that each of its internal node, labelled as A say, has two children B_1 and B_2 , such that A is a subset of $[n]$ and $A = B_1 \sqcup B_2$; this splitting of A into two subsets is caused by the removal of an edge (i.e. a cut) in T^{PF} ; we then label the two edges pointing towards A in $\text{cut}(T^{\text{PF}})$ with the same label of this removed edge in T^{PF} (see Fig. 5). Observe that if an edge is labelled as $n - k$, then it corresponds to the k -th cut in the fragmentation of T^{PF} . To show (48), we first notice that

$$d^{\text{SC}}(v, v') = H^{\text{SC}}(v) + H^{\text{SC}}(v') - 2H^{\text{SC}}(b),$$

where b is the branch point between v and v' in T^{SC} , namely, the unique node of T^{SC} such that v, v' and the root are in three separate components of $T^{\text{SC}} \setminus \{b\}$. Similarly, we have

$$d^{\text{cut}}(v, v') = H^{\text{cut}}(v) + H^{\text{cut}}(v') - 2H^{\text{cut}}(B),$$

where B is the branch point between $\{v\}$ and $\{v'\}$ in $\text{cut}(T^{\text{PF}})$ and $H^{\text{cut}}(B)$ stands for the height of B . We claim

$$H^{\text{SC}}(b) \leq H^{\text{cut}}(B) + 1 \leq X^{\text{SC}}(b). \quad (49)$$

Proof of (49): Let $n - k_B$ be the label of the two in-going edges of B in $\text{cut}(T^{\text{PF}})$. By the way we construct $\text{cut}(T^{\text{PF}})$ and define its edge labels, we have $k_B = \inf\{k : \pi_{n-k}^{\text{PF}}(v) \neq \pi_{n-k}^{\text{PF}}(v')\}$, namely, the index of the cut which separates v and v' . It follows that $H^{\text{cut}}(B) + 1 = \#\{1 \leq k \leq k_B : \pi_{n-k}^{\text{PF}}(v) \neq \pi_{n-k-1}^{\text{PF}}(v)\}$. On the other hand, let $n - k_b$ be the label of the out-going edge of b . Notice that after each removal of an edge on the path in T^{SC} from the root to b , the vertices v and v' are still in the same tree component of some forest F_{n-k}^{SC} . Also recall that $\Pi_{n-k}^{\text{PF}} = \Pi_{n-k}^{\text{SC}}$ for all $1 \leq k \leq n - 1$ in our coupling. Thus, $k_b \leq k_B$ and then $H^{\text{SC}}(b) = \#\{1 \leq k \leq k_b : \Pi_{n-k}^{\text{SC}}(v) \neq \Pi_{n-k-1}^{\text{SC}}(v)\} \leq H^{\text{cut}}(B) + 1$. This is the first inequality in (49). For the second one, let $n - k_v$ (resp. $n - k_{v'}$) be the label of the first edge on the path from b to v (resp. v') in T^{SC} , if $v \neq b$ (resp. $v' \neq b$); otherwise, set $k_v = k_b$ (resp. $k_{v'} = k_b$). Without loss of generality, we assume that $k_v \geq k_{v'}$. Observe that after removal of this edge, v and v' are separated. Thus, $k_v \geq k_B$. Recall

b is separated in the fragmentation of T^{SC} only after all the edges on the path between it and the root and also all its in-going edges (thus including the edge labelled $k_{v'}$) are removed. It follows

$$X^{\text{SC}}(b) \geq \#\{1 \leq k \leq k_v : \pi_{n-k}^{\text{SC}}(v) \neq \pi_{n-k-1}^{\text{SC}}(v)\} \geq \{k \leq k_B : \pi_{n-k}^{\text{PF}}(v) \neq \pi_{n-k}^{\text{PF}}(v')\} = H^{\text{cut}}(B) + 1.$$

This proves (49).

Combining (49) with (40) and (44), we see that

$$n^{-\frac{1}{2}} \cdot |H^{\text{SC}}(b) - H^{\text{cut}}(B)| \leq n^{-\frac{1}{2}} D^{\text{SC}}(b) \xrightarrow[n \rightarrow \infty]{\mathbb{P}} 0.$$

By previous arguments, this is enough to conclude the proof. \square

Proposition 26 suggests that T^{SC} and $\text{cut}(T^{\text{PF}})$, with their graph distances rescaled by $\frac{1}{\sqrt{n}}$, should converge to the same limit, if such a limit exists (since the asymptotics of the leaves determines the asymptotics of every nodes, when the limit is the Brownian CRT). This is indeed shown by Bertoin & Miermont [10]; moreover, they show that the scaling limit of $\text{cut}(T^{\text{PF}})$ is the so-called cut tree of the Brownian CRT. A formal definition of this object requires certain amount of preliminaries on the CRT theory. Here we only give an informal description and refer the interested readers to Introduction of [10]. Recall that $\frac{1}{\sqrt{n}}T^{\text{PF}}$ converges in distribution to Aldous' Brownian CRT \mathcal{T} . The convergence takes place with respect to the so-called Gromov–Hausdorff topology [17] on the space of compact metric spaces. Furthermore to this convergence of trees, the previous edge-cutting process on T^{PF} , once suitably rescaled both in space and in time, converges to a random cutting process of \mathcal{T} ([5]). Bertoin and Miermont develop the idea of (discret) cut tree and introduce a similar notion of cut tree for the Brownian CRT \mathcal{T} , which encodes in a certain way the genealogy of the fragmentation induced by the random cutting on \mathcal{T} . Moreover, their result (Theorem 1 of [10]) entails that

$$\left(\frac{1}{\sqrt{n}}T^{\text{PF}}, \frac{1}{\sqrt{n}}\text{cut}(T^{\text{PF}}) \right) \xrightarrow[n \rightarrow \infty]{(d)} (\mathcal{T}, \text{cut}(\mathcal{T})), \quad (50)$$

where $\text{cut}(\mathcal{T})$ stands for the cut tree of \mathcal{T} and the convergence of each coordinate is with respect to the Gromov–Hausdorff topology.

Remark on (50). There is a slight difference in the definitions of $\text{cut}(T^{\text{PF}})$ in [9] and [10]. However, by comparing the distributions of the subtrees of $\text{cut}(T^{\text{PF}})$ spanned by k uniform vertices in both cases, for all $k \geq 1$, we easily see that the difference is negligible as $n \rightarrow \infty$. On the other hand, the convergence in Theorem 1 of [10] is stated for the Gromov–Prokhorov topology in general. However, in the current case, Bertoin [9] has previously shown that $\frac{1}{\sqrt{n}}\text{cut}(T^{\text{PF}})$ converges in Gromov–Hausdorff sense to the Brownian CRT, therefore its distribution is tight in the weak topology for the Gromov–Hausdorff metric. Then Theorem 1 of [10], combined with a habituel argument, confers that the limit point is unique and thus (50) holds true.

Now, from the convergence (48) in Proposition 26, it is not difficult to show the following:

$$\left(\left(\frac{1}{\sqrt{n}}T^{\text{PF}}, \frac{1}{\sqrt{n}}T^{\text{SC}} \right), n \geq 1 \right) \xrightarrow[n \rightarrow \infty]{(d)} (\mathcal{T}, \text{cut}(\mathcal{T})), \quad (51)$$

where the convergence of each coordinate is with respect to the Gromov–Hausdorff topology. However, we do not wish to develop the arguments here, as that will inevitably involve technical points on the Gromov–Hausdorff topology.

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